MATH 3B (Butler)
Practice for Midterm II (Solutions)

1. Find $\int_0^e \ln x\,dx$. (Hint: L'Hospital's rule might be useful.)

Before we start we note that $\ln x$ has a vertical asymptote at $x = 0$ and so this is an improper integral. First we find the indefinite integral of $\ln x$, using integration by parts we have

$$\underbrace{\int \ln x}_{\substack{u=\ln x \\ du=\frac{1}{x}\,dx} \quad \substack{dv=dx \\ v=x}} = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x + C.$$

So we have

$$\begin{aligned}
\int_0^e \ln x\,dx &= \lim_{c\to 0^+}\left(\int_c^e \ln x\,dx\right) \\
&= \lim_{c\to 0^+}\left((x\ln x - x)\Big|_c^e\right) \\
&= \lim_{c\to 0^+}\Big((e\ln e - e) - (c\ln c - c)\Big) \\
&= -\lim_{c\to 0^+} c\ln c \\
&= -\lim_{c\to 0^+}\frac{\ln c}{1/c} \\
&= -\lim_{c\to 0^+}\frac{1/c}{-1/c^2} \\
&= \lim_{c\to 0^+} c \;=\; 0
\end{aligned}$$

The last part coming from L'Hospital's rule (it turned out that it was useful).

2. (a) Find $\int \frac{dx}{e^{2x}-1}$ by substituting $u = e^x$.

If we use the substitution $u = e^x$ then we have $du{=}e^x\,dx{=}u$ or $dx{=}\frac{1}{u}\,du$. We also have that $e^{2x} = (e^x)^2 = u^2$.

$$\int \frac{dx}{e^{2x}-1} = \int \frac{du}{u(u^2-1)} = \int \frac{du}{u(u-1)(u+1)}$$

This is an integration by parts problem. So next we break it up as follows

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1},$$

clearing the denominators we get

$$1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1).$$

Choosing some convenient values for u we see $A = -1$ $(u = 0)$, $B = \frac{1}{2}$ $(u = 1)$ and $C = \frac{1}{2}$ $(u = -1)$. So we have

$$\begin{aligned} \int \frac{du}{u(u^2-1)} &= \int \frac{du}{u(u-1)(u+1)} = \int \left(\frac{-1}{u} + \frac{1}{2}\frac{1}{u-1} + \frac{1}{2}\frac{1}{u+1} \right) du \\ &= -\ln|u| + \frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1| + C \end{aligned}$$

Finally substituting back $u = 2x$ we see

$$\begin{aligned} \int \frac{dx}{e^{2x}-1} &= -\ln|e^x| + \frac{1}{2}\ln|e^x-1| + \frac{1}{2}\ln|e^x+1| + C \\ &= -x + \ln\sqrt{|e^{2x}-1|} + C. \end{aligned}$$

(b) Find $\int \frac{dx}{e^{2x}-1}$ by substituting $u = e^{2x} - 1$.

If we use the substitution $u = e^{2x} - 1$ then we have $du = 2e^{2x}\,dx = 2(u+1)\,dx$ or $dx = \frac{1}{2(u+1)}\,du$.

$$\int \frac{dx}{e^{2x}-1} = \int \frac{du}{2u(u+1)}.$$

This is an integration by parts problem. So next we break it up as follows

$$\frac{1/2}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1},$$

clearing the denominators we get

$$\frac{1}{2} = A(u+1) + Bu.$$

Choosing some convenient values for u we see $A = \frac{1}{2}$ $(u = 0)$ and $B = -\frac{1}{2}$ $(u = -1)$. So we have

$$\int \frac{du}{2u(u+1)} = \int \left(\frac{1}{2}\frac{1}{u} - \frac{1}{2}\frac{1}{u+1}\right) du = \frac{1}{2}\ln u - \frac{1}{2}\ln|u+1| + C$$

Finally, substituting back $u = e^{2x} - 1$ we see

$$\int \frac{dx}{e^{2x}-1} = \frac{1}{2}\ln|e^{2x}-1| - \frac{1}{2}\ln|e^{2x}| + C = \ln\sqrt{|e^{2x}-1|} - x + C.$$

3. The function $f(x)$ has
$$P_2(x) = \frac{1}{3} + 2x - \frac{1}{4}x^2$$
as its degree 2 Taylor polynomial around $x = 0$. Let $g(x) = \arctan(\sqrt{3}f(x))$. Find the degree 2 Taylor polynomial for $g(x)$ around $x = 0$. (Hint: $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$, $\arctan(1) = \frac{\pi}{4}$ and $\arctan(\sqrt{3}) = \frac{\pi}{3}$.)

First we note that by the definition of Taylor polynomials that
$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = \frac{1}{3} + 2x - \frac{1}{4}x^2,$$
and in particular we see that $f(0) = \frac{1}{3}$, $f'(0) = 2$ and $f''(0) = -\frac{1}{2}$. We are looking for the degree 2 Taylor polynomial for $g(x)$ around $x = 0$ which is given by
$$\widehat{P}_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2,$$
so we need to calculate these values for g given what we know about f. In particular we have (using the chain rule and the quotient rule):
$$\begin{aligned} g(x) &= \arctan(\sqrt{3}f(x)) \\ g'(x) &= \frac{\sqrt{3}f'(x)}{1 + 3(f(x))^2} \\ g''(x) &= \frac{(1 + 3(f(x))^2)\sqrt{3}f''(x) - \sqrt{3}f'(x) \cdot 6f(x)f'(x)}{(1 + 3(f(x))^2)^2} \end{aligned}$$
Evaluating at 0 and substituting in what we know about f we have
$$\begin{aligned} g(0) &= \arctan(\sqrt{3} \cdot \frac{1}{3}) = \frac{\pi}{6} \\ g'(0) &= \frac{\sqrt{3} \cdot 2}{1 + 3(\frac{1}{3})^2} = \frac{3}{2}\sqrt{3} \\ g''(0) &= \frac{(1 + 3(\frac{1}{3})^2)\sqrt{3} \cdot (-\frac{1}{2}) - \sqrt{3} \cdot 2 \cdot 6 \cdot \frac{1}{3} \cdot 2}{(1 + 3(\frac{1}{3})^2)^2} = -\frac{39}{8}\sqrt{3} \end{aligned}$$
Putting these values in we get our desired Taylor polynomial,
$$\widehat{P}_2(x) = \frac{\pi}{6} + \frac{3}{2}\sqrt{3}x - \frac{39}{16}\sqrt{3}x^2.$$

4. Now, the Star-Belly Sneetches had bellies with stars. The Plain-Belly Sneetches had none upon thars. Then one day Sylvester McMonkey McBean came to town with his wondrously wonderful machine, "Just one pass through, hop on board, and you will have a star for sure." The Sneetches listened and the Sneetches thought and those who wanted a star-belly stepped up and bought.

Sylvester kept track of the percentage of Sneetches with stars (Q) and noticed with time (t) in months that

$$\frac{dQ}{dt} = \frac{1}{3}(1-Q)^2 t.$$

His business was quick, he did not want to delay, and so he recalled on his very first day that $Q = \frac{1}{4}$, to make a quick buck and then leave this place he decided to leave when $Q = \frac{3}{4}$. How many months then will it take until Sylvester McMonkey McBean leaves this place?

We are dealing with a separable differential equation so as a first step we separate and get

$$\frac{dQ}{(1-Q)^2} = \frac{1}{3}t\,dt.$$

Integrating both sides we have

$$\frac{1}{1-Q} = \frac{1}{6}t^2 + C,$$

using that initially $Q = 1/4$ (i.e., $Q(0) = 1/4$) we have

$$\frac{1}{1-\frac{1}{4}} = \frac{1}{6}0^2 + C \quad \text{or} \quad C = \frac{4}{3}.$$

Putting in the value for C we now have

$$\frac{1}{1-Q} = \frac{1}{6}t^2 + \frac{4}{3}.$$

We *could* solve for Q (but we weren't asked to so why do any extra work), instead let us now put in $Q = 3/4$ and solve for t getting our desired answer. In particular we want t so that

$$\frac{1}{1-\frac{3}{4}} = \frac{1}{6}t^2 + \frac{4}{3} \quad \text{or rearranging} \quad 16 = t^2.$$

So we see that our solution is $t = 4$ (we discount the $t = -4$ solution since it does not fit the framework of our problem). So we see that Sylvester McMonkey McBean will leave after four months.

5. Find $\int e^{\sqrt[3]{x}}\,dx$.

We begin by making the substitution $t = \sqrt[3]{x} = x^{1/3}$ so that $dt = \frac{1}{3}x^{-2/3}\,dx$ or $dx = 3x^{2/3}\,dt = 3t^2\,dt$. After making this substitution this problem then becomes a straightforward integration by parts (done twice). So we have

$$\begin{aligned}
\int e^{\sqrt[3]{x}}\,dx &= \underbrace{\int 3t^2 e^t\,dt}_{\substack{u=3t^2 \quad dv=e^t\,dt \\ du=6t\,dt \quad v=e^t}} \\
&= 3t^2e^t - \underbrace{\int 6te^t\,dt}_{\substack{u=6t \quad dv=e^t\,dt \\ du=6\,dt \quad v=e^t}} \\
&= 3t^2e^t - \left(6te^t - \int 6e^t\,dt\right) \\
&= 3t^2e^t - 6te^t + 6e^t + C \\
&= 3x^{2/3}e^{x^{1/3}} - 6x^{1/3}e^{x^{1/3}} + 6e^{x^{1/3}} + C.
\end{aligned}$$