MATH 3B (Butler)
Practice for Final (II, Solutions)

1. Let $G(x) = \int_{\sin x}^{x^2} e^{t^2}\,dt$. Find the degree 2 Taylor polynomial about $x = 0$ for $G(x)$.

The degree 2 Taylor polynomial is given by

$$P_2(x) = G(0) + G'(0)x + \frac{G''(0)}{2}x^2.$$

We now need to calculate $G(0)$, $G'(0)$ and $G''(0)$. We have

$$G(0) = \int_0^0 e^{t^2}\,dt = 0,$$

since the upper and lower bounds match. Next, by Leibniz's rule we have

$$G'(x) = \frac{d}{dx}\left(\int_{\sin x}^{x^2} e^{t^2}\,dt\right) = e^{(x^2)^2}(2x) - e^{(\sin x)^2}(\cos x) = 2xe^{x^4} - \cos x e^{\sin^2 x}$$

and so $G'(0) = -1$. Finally we have

$$G''(x) = 2e^{x^4} + 8x^4e^{x^4} + (\sin x)e^{\sin^2 x} - 2\sin x \cos^2 x e^{\sin^2 x}$$

and so $G''(0) = 2$.
Putting this altogether we have

$$P_2(x) = -x + x^2.$$

2. For y between 0 and 4 find the area between the curves $g(y) = 4y^2 + 8y + 23$ and $h(y) = -2y^2 + 26y + 23$.

First we check for intersections. We have

$$4y^2 + 8y + 23 = -2y^2 + 26y + 23 \quad \text{or} \quad 6y^2 - 18y = 0 \quad \text{or} \quad 6y(y-3) = 0.$$

So they intersect at 0 and 3, so to find the area we need to break it up to finding the area between 0 and 3 and also the area between 3 and 4. By looking at the functions it is easy to check that $h(1) \geq g(1)$ so that $h(y) \geq g(y)$ for $0 \leq y \leq 3$ and $g(y) \geq h(y)$ for $y \leq 0$ and $y \geq 3$. So we now have

$$\begin{aligned}
\text{Area} &= \int_0^3 (h(y) - g(y))\, dy + \int_3^4 (g(y) - h(y))\, dy \\
&= \int_0^3 (18y - 6y^2)\, dy + \int_3^4 (6y^2 - 18y)\, dy \\
&= (9y^2 - 2y^3)\Big|_0^3 + (2y^3 - 9y^2)\Big|_3^4 \\
&= ((81 - 54) - (0 - 0)) + ((128 - 144) - (54 - 81)) \\
&= 38.
\end{aligned}$$

3. Find $\int_{1/2}^{1} r(4x)\,dx$ given

$$\int_1^3 r(2x)\,dx = 5, \qquad \int_1^2 r(3x)\,dx = 4, \qquad \int_4^3 r(x)\,dx = 2 \quad \text{and} \quad \int_0^1 r(5x)\,dx = 6.$$

First let us translate all of these integrals by using substitution so that they are all comparable. For example, if we let $u = 4x$ we have $du = 4\,dx$ and so we are trying to find

$$\int_{1/2}^{1} r(4x)\,dx = \frac{1}{4}\int_2^4 r(u)\,du.$$

Repeating this with the other integrals (along with swapping order gives a negative sign) we have that

$$\int_2^6 r(u)\,du{=}10, \quad \int_3^6 r(u)\,du{=}12, \quad \int_3^4 r(u)\,du{=}-2 \quad \text{and} \quad \int_0^5 r(u)\,du{=}30.$$

Now we just have to find a way to combine all of these integrals to give us the integral from 2 to 4, this is not too hard. We now have

$$\begin{aligned} \frac{1}{4}\int_2^4 r(u)\,du &= \frac{1}{4}\Big(\int_2^6 r(u)\,du - \int_4^6 r(u)\,du\Big) \\ &= \frac{1}{4}\Big(\int_2^6 r(u)\,du - \Big(\int_3^6 r(u)\,du - \int_3^4 r(u)\,du\Big)\Big) \\ &= \frac{1}{4}(10 - 12 - 2) \\ &= -1. \end{aligned}$$

4. Find $\displaystyle\int_0^\infty \frac{\sqrt{y}}{y^3+1}\,dy$.

(Hint: you do **not** need to know that $y^3+1=(y+1)(y^2-y+1)$.)

Now this is a weird hint, why couldn't we get something about what we are supposed to do. So I gather from the hint that we are not trying to use the method of partial fractions, but we couldn't anyways since this is **not** a rational function. The problem is that we don't have polynomials, so let us perhaps try rewriting it using substitution. We see that we have a term $\sqrt{y}$, so we want to make a substitution that helps to get rid of it. There are several that work, the second simplest is $u=y^{3/2}$ so that $du=\frac{3}{2}y^{1/2}\,dy$. (The simplest is $u=y^{1/2}$ and this substitution works but requires a later second substitution so we are saving one step by going second best!) Making this substitution we now have (for the indefinite integral)

$$\int \frac{\sqrt{y}}{y^3+1}\,dy=\frac{2}{3}\int\frac{1}{u^2+1}\,du=\frac{2}{3}\arctan u+C=\frac{2}{3}\arctan\left(y^{3/2}\right)+C.$$

Now we are ready to turn to the fact that we are dealing with an indefinite integral, so we now have

$$\begin{aligned}\int_0^\infty \frac{\sqrt{y}}{y^3+1}\,dy &= \lim_{t\to\infty}\int_0^t \frac{\sqrt{y}}{y^3+1}\,dy\\ &= \lim_{t\to\infty}\left(\frac{2}{3}\arctan\left(y^{3/2}\right)\right)\Big|_0^t\\ &= \frac{2}{3}\lim_{t\to\infty}\arctan\left(t^{3/2}\right)\\ &= \frac{\pi}{3}.\end{aligned}$$

In the last step we used that as $t\to\infty$ the function $\arctan(t^{3/2})$ approaches the horizontal asymptote $y=\pi/2$.

5. (a) Use substition to show that for a function $f(x)$ that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx.$$

If we make the substitution $u = a - x$, or $x = a - u$ then we have $du = -dx$ and so

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,du = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du$$

Since the u is a "dummy variable" we can replace all the u's in the last integral by x's and we get the result.

(b) Show that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{1}{2}a.$$

(Hint: $\int_0^a g(x)\,dx = \frac{1}{2}\left(\int_0^a g(x)\,dx + \int_0^a g(x)\,dx\right)$, and use part (a).)

Using the hint we have (along with the previous part)

$$\begin{aligned}
\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx &= \frac{1}{2}\left(\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx\right) \\
&= \frac{1}{2}\left(\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx\right) \\
&= \frac{1}{2}\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx \\
&= \frac{1}{2}\int_0^a 1\,dx \;=\; \frac{1}{2}a.
\end{aligned}$$

(c) Find $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$. (Hint: $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.)

We first rewrite this and using the hint we have

$$\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \cos\left(\frac{\pi}{2} - x\right)}\,dx.$$

This is the integral done in part (b) where we have $a = \pi/2$ and $f(x) = \cos x$; so we can conclude that the integral is $\pi/4$.

6. Find $\int \frac{e^x - 1}{e^x + 1}\, dx$.

Let us first rewrite it by making the substitution $u = e^x$ so that $du = e^x\, dx = u\, dx$ or $dx = du/u$. In particular, we have

$$\int \frac{e^x - 1}{e^x + 1}\, dx = \int \frac{u - 1}{u(u+1)}\, du.$$

In this form we see that this is a straightforward integration using the method of partial fractions. So we first check and see that no long division is required. So the next step is for us to break it into smaller parts, namely

$$\frac{u-1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

Clearing the denominators this is the same as $u - 1 = A(u+1) + Bu$, we can now either group coefficients of by choosing "nice" values of u we see that $A = -1$ $(u = 0)$ and $B = 2$ $(u = -1)$. So we now have

$$\int \frac{u-1}{u(u+1)}\, du = \int \Big(-\frac{1}{u} + 2\frac{1}{u+1} \Big)\, du = -\ln u + 2\ln(u+1) + C.$$

So we have

$$\int \frac{e^x - 1}{e^x + 1}\, dx = -\ln e^x + 2\ln(e^x + 1) + C = -x + 2\ln(e^x + 1) + C$$

7. Consider the autonomous differential equation

$$\frac{dy}{dt} = \frac{1 - y^2}{1 + y^2}.$$

(a) Find the equilibrium solutions to the differential equation, and determine the stability of each equilibrium point.

The equilibrium solutions. or constant solutions are found by solving

$$\frac{1 - y^2}{1 + y^2} = 0 \quad \text{or} \quad y^2 = 1.$$

So the equilibrium solutions are at $y = \pm 1$.
To determine their stability we can draw a graph or test the "eigenvalues". In the latter case if we let

$$g(y) = \frac{1 - y^2}{1 + y^2} \quad \text{then} \quad g'(y) = \frac{(1 + y^2)(-2y) - (1 - y^2)(2y)}{(1 + y^2)^2} = -\frac{4y}{(1 + y^2)^2}.$$

In particular we see that $g(-1) = 1 > 0$ and so -1 is an unstable equilibrium solution, while $g(1) = -1 < 0$ and so 1 is a stable equilibrium solution.

(b) If the initial condition $y(0) = 2$ is given, what happens to y as $t \to \infty$? (Hint: you do not need to solve the differential equation to answer this part.)

We are starting at $y = 2$, by looking at $g(y)$ we see that for $y > 1$ that $g(y) < 0$. This shows that for $y > 1$ that $dy/dt < 0$ and in particular the value of y will decrease. And so the solution will be decreasing and continue to decrease towards the equilibrium solution $y = 1$, i.e., as $t \to \infty$ we have that $y \to 1$.

8. (a) Find an equation for the line in parametric form which passes through the point $(2, -1, 4)$ and is perpendicular to the plane $x - 3y + 4z = -5$.

> To describe a line we need a point (in this case we are given $(2, -1, 4)$) and a vector to describe the direction. In this case we have that the line must be perpendicular to the plane. We know that one vector which is perpendicular to the plane is the normal vector, and this can easily be found by reading off the coefficients we have that the normal vector is $[1, -3, 4]'$, so this gives us the vector describing the direction of the line. So we now have that the line is
>
> $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \text{or} \quad \begin{array}{rcl} x & = & 2 + t \\ y & = & -1 - 3t \\ z & = & 4 + 4t \end{array} .$$

(b) Find the point in the plane $x - 3y + 4z = -5$ closest to the point $(2, -1, 4)$. (Hint: this point *must* lie on the line found in part (a), so we need to find the right "t" so that the point in the line is also in the plane.)

> This hint makes the problem much easier. Really, we are trying to find where the line hits the plane. We know that points on the plane satisfy $x - 3y + 4z = -5$ and so for the line to intersect the plane we must have (using what we found in (a)):
>
> $$-5 = (2 + t) - 3(-1 - 3t) + 4(4 + 4t) = 21 + 26t \quad \text{or} \quad t = -1.$$
>
> So the line intersects the plane when $t = -1$ which corresponds to the point $(1, 2, 0)$, and this is the point in the plane closest to $(2, -1, 4)$.

9. Find $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x + x^2y^2}$. (Hint: break the function into two pieces.)

Note: technically we should also define what the function should be for $x = 0$, ignore this technicality.

We note that we can rewrite the function inside the limit as

$$\Big(\underbrace{\frac{\sin x}{x}}_{=f(x)}\Big)\Big(\underbrace{\frac{e^y}{1 + xy^2}}_{=g(x)}\Big).$$

Now if we know the limit of each piece then we can use the fact that $\lim\,(f(x)g(x)) = (\lim f(x))(\lim g(x))$ to evaluate the limit. In our case, each piece is easy to deal with. Namely we have

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

(a well known limit from Math 3a), while

$$\lim_{(x,y)\to(0,0)} \frac{e^y}{1 + xy^2} = 1$$

since this is continuous at $(0,0)$ and we can find it by just evaluating the function at $(0,0)$. So combining we can conclude that

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x + x^2y^2} = 1$$

10. One way to describe a surface is as $z = f(x, y)$, but we can also describe a surface by $F(x, y, z) = c$, where c is a fixed constant. The tangent plane of $F(x, y, z) = c$ at the point (x_0, y_0, z_0) is given by

$$\frac{\partial F(x_0, y_0, z_0)}{\partial x}(x - x_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial y}(y - y_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial z}(z - z_0) = 0$$

Find the tangent plane to the ellipsoid (the three dimensional analog of an ellipse) $x^2 + 4y^2 + 9z^2 = 34$ at the point $(3, -2, -1)$.

This question is really not about tangent planes, but is really about partial derivatives and seeing if we know how to follow directions. First we compute the partial derivatives, in our case we have $F(x, y, z) = x^2 + 4y^2 + 9z^2$ and the point $(x_0, y_0, z_0) = (3, -2, -1)$. So we have

$$\begin{aligned} \frac{\partial F(x,y,z)}{\partial x} &= 2x \quad \text{so} \quad \frac{\partial F(3,-2,-1)}{\partial x} = 6 \\ \frac{\partial F(x,y,z)}{\partial y} &= 8y \quad \text{so} \quad \frac{\partial F(3,-2,-1)}{\partial y} = -16 \\ \frac{\partial F(x,y,z)}{\partial z} &= 18z \quad \text{so} \quad \frac{\partial F(3,-2,-1)}{\partial z} = -18 \end{aligned}$$

We now have all the terms in the tangent plane formula given us and so we have that the tangent plane is

$$6(x - 3) - 16(y + 2) - 18(z + 1) = 0 \quad \text{or} \quad 3x - 8y - 9z = 34.$$

11. Find the gradient for $g(x, y) = 3x^2y - \sin(xy) + e^y - \ln(x^2 + 1)$.

We have

$$\nabla g(x, y) = \begin{bmatrix} \dfrac{\partial g(x,y)}{\partial x} \\ \dfrac{\partial g(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 6xy - y\cos(xy) - \dfrac{2x}{x^2+1} \\ 3x^2 - x\cos(xy) + e^y \end{bmatrix}.$$

(This almost seems too easy!)

12. You have been hired to design a new shipping container. The design is simple enough, a rectangular box, so your main job is to decide what dimension to make the box. The materials to make the top/bottom sides cost $\$2/m^2$ the materials to make the front/back sides cost $\$3/m^2$ and the material to make the left/right sides cost $\$5/m^2$. If the container must hold $240\,m^3$, find the dimensions of the box that minimize the cost.

First we translate this into a more mathematical form. There are three dimensions for the box which we will denote by length by x, width by y and height by z. The area of a top/bottom side is xy, the area of a front/back side is xz and the area of a left/right side is xz. So then using the cost of material for the sides (and remembering we have two of each type of side) the total cost of the shipping container is

$$f(x, y, z) = 4xy + 6xz + 10yz.$$

We also have that the container must hold $240\,m^3$ so that tells us that $xyz = 240$, this acts as our constraint. So we now see that we are maximizing a function $f(x, y, z)$ given $g(x, y, z) = c$, a Lagrange multiplier problem. So we need to find the critical points for

$$F(x, y, z, \lambda) = 4xy + 6xz + 10yz - \lambda(xyz - 240).$$

We have

$$\begin{aligned}
\frac{\partial F(x, y, z, \lambda)}{\partial x} &= 4y + 6z - \lambda yz = 0 \\
\frac{\partial F(x, y, z, \lambda)}{\partial y} &= 4x + 10z - \lambda xz = 0 \\
\frac{\partial F(x, y, z, \lambda)}{\partial z} &= 6x + 10y - \lambda xy = 0 \\
\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} &= -xyz + 240 = 0
\end{aligned}$$

We now can solve the first three equations for λ and doing so we can conclude that

$$\frac{4}{z} + \frac{6}{y} = \frac{4}{z} + \frac{10}{x} = \frac{6}{y} + \frac{10}{x}.$$

From this it is easy to see that $6/y = 10/x$ or $y = (3/5)x$ and $4/z = 10/x$ or $z = (2/5)x$. Finally using the last equation we have

$$240 = xyz = x(\frac{3}{5}x)(\frac{2}{5}x) = \frac{6}{25}x^3 \quad \text{or} \quad x^3 = 1000 \quad \text{or} \quad x = 10.$$

So the cost minimizing dimensions are $x = 10\,m$, $y = 6\,m$ and $z = 4\,m$.