MATH 3B (Butler)
Practice for Final (I, Solutions)

1. Gabriel's horn is a mathematical object taken by rotating the curve $y = \frac{1}{x}$ around the x-axis for $1 \le x < \infty$. It can be shown (though not by what we have done) that the resulting *surface* of the horn has infinite area, i.e., it is impossible to paint the outside of the horn. Find the volume of Gabriel's horn (i.e., find how much paint it would take to fill the inside of the horn).

This is finding the volume of revolution between the curve $y = \frac{1}{x}$ and the x-axis from 1 to ∞. Setting this up as an integral we have

$$\text{Volume} = \pi \int_1^\infty (\frac{1}{x})^2 \, dx.$$

This is an improper integral and so we use the techniques of improper integration to find the answer. In particular, we have

$$\begin{aligned} \text{Volume} &= \lim_{t\to\infty} \left(\pi \int_1^t x^{-2} \, dx \right) \\ &= \pi \lim_{t\to\infty} \left(-x^{-1} \Big|_1^t \right) \\ &= \pi \lim_{t\to\infty} \left(-\frac{1}{t} + 1 \right) \\ &= \pi. \end{aligned}$$

(Note: this shows that even though there is not enough paint to paint the surface of Gabriel's horn there is enough paint for us to fill up the whole horn, a fun mathematical paradox.)

2. Reduce the following to a single integral of the form $A\int_B^C f(x)\,dx$ for some constants A, B, C.

$$\int_0^5 f(x)\,dx - \int_3^3 f(e^x)\,dx + \int_0^1 3f(3x)\,dx - \int_0^4 f(\frac{1}{2}x)\,dx + \int_5^3 f(x)\,dx.$$

We first note that $\int_3^3 f(e^x)\,dx = 0$ since the ending and starting points are the same. We also have that $\int_5^3 f(x)\,dx = -\int_3^5 f(x)\,dx$ by a basic property of integrals. Using substitution ($u = 3x$ so $du = 3\,dx$) we have $\int_0^1 3f(3x)\,dx = \int_0^3 f(x)\,dx$, similarly we have $\int_0^4 f(\frac{1}{2}x)\,dx = 2\int_0^2 f(x)\,dx$. Putting all of this together we see that we can rewrite out expression as

$$\int_0^5 f(x)\,dx - 0 + \int_0^3 f(x)\,dx - 2\int_0^2 f(x)\,dx - \int_3^5 f(x)\,dx.$$

We can combine the first and last piece as follows

$$\int_0^5 f(x)\,dx - \int_3^5 f(x)\,dx = \int_0^3 f(x)\,dx$$

so that we now have

$$2\int_0^3 f(x)\,dx - 2\int_0^2 f(x)\,dx.$$

Finally, we can combine these two similarly as we did before to get

$$2\int_2^3 f(x)\,dx.$$

3. Find the average value for $f(x) = |x^2 - 4|$ for the interval $-3 \le x \le 5$.

The average value is found by

$$\frac{1}{5-(-3)} \int_{-3}^{5} |x^2 - 4| \, dx.$$

What makes this hard of course is the absolute value, so what we do is break it up into pieces. The zeroes of $x^2 - 4$ are at $x = -2$ ad $x = 2$ and so we have

$$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x \le -2, \\ 4 - x^2 & \text{if } -2 \le x \le 2, \\ x^2 - 4 & \text{if } x \ge 2. \end{cases}$$

Using this we can break our integral into three pieces, so that our average value is

$$\begin{aligned} \frac{1}{8}\int_{-3}^{5} |x^2 - 4| \, dx &= \frac{1}{8}\Big(\int_{-3}^{-2} (x^2 - 4) \, dx + \int_{-2}^{2} (4 - x^2) \, dx + \int_{2}^{5} (x^2 - 4) \, dx\Big) \\ &= \frac{1}{8}\Big((\frac{1}{3}x^3 - 4x)\Big|_{-3}^{-2} + (4x - \frac{1}{3}x^3)\Big|_{-2}^{2} + (\frac{1}{3}x^3 - 4x)\Big|_{2}^{5}\Big) \\ &= \frac{1}{8}\Big((\frac{16}{3} - 3) + (\frac{16}{3} + \frac{16}{3}) + (\frac{65}{3} + \frac{16}{3})\Big) \\ &= 5. \end{aligned}$$

4. Given $g(0) = 3$, $g'(0) = -1$, $g''(0) = \pi$, $g(1) = 5$, $g'(1) = 2$ and $g''(1) = 1$, find

$$\int_0^1 x^2 g'''(x)\,dx.$$

Using integration by parts we have

$$\begin{aligned}
\underbrace{\int_0^1 x^2 g'''(x)\,dx}_{\substack{u=x^2 \\ du=2x\,dx} \quad \substack{v=g''(x) \\ dv=g'''(x)\,dx}} &= x^2 g''(x)\Big|_0^1 - \underbrace{\int_0^1 2xg''(x)\,dx}_{\substack{u=2x \\ du=2\,dx} \quad \substack{v=g'(x) \\ dv=g''(x)\,dx}} \\
&= x^2 g''(x)\Big|_0^1 - \left(2xg'(x) - \int_0^1 2g'(x)\,dx\right) \\
&= x^2 g''(x)\Big|_0^1 - 2xg'(x)\Big|_0^1 + 2g(x)\Big|_0^1 \\
&= (1-0) - (4-0) + (10-6) \\
&= 1.
\end{aligned}$$

5. Find $\int \frac{2\ln(x+1)}{x^3}\,dx$.

At first glance this integral looks terrible. But we know that $\ln(x+1)$ has a nice derivative and so let us first try integration by parts.

$$\underbrace{\int \frac{2\ln(x+1)}{x^3}\,dx}_{\substack{u=\ln(x+1) \quad v=-\frac{1}{x^2} \\ du=\frac{1}{x+1}\,dx \quad dv=\frac{2}{x^3}\,dx}} = -\frac{\ln(x+1)}{x^2} + \int \frac{1}{x^2(x+1)}\,dx$$

Now, this is an integral that we now how to do something with, namely partial fractions. So as a first step we need to decompose the terms as follows

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

Clearing the denominator we have

$$1 = Ax(x+1) + B(x+1) + Cx^2,$$

by choosing $x = 0$ we see that $B = 1$ and by choosing $x = -1$ we see that $C = 1$. Finally by choosing $x = 1$ we see that $1 = 2A + 2B + C = 2A + 3$ so that $A = -1$. Putting this in we have

$$\begin{aligned} \int \frac{2\ln(x+1)}{x^3}\,dx &= -\frac{\ln(x+1)}{x^2} + \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx \\ &= -\frac{\ln(x+1)}{x^2} - \ln|x| - \frac{1}{x} + \ln|x+1| + C. \end{aligned}$$

6. Scientists have recently discovered a new species of tree, *gumdropus delectus*, which aside from its unusually sticky fruit has also been shown to exhibit some unusual growth pattern. Namely, the height H of the tree (measured in feet) satisfies the differential equation

$$\frac{dH}{dt} = \frac{1}{100}(40 - H)^2,$$

where t is time measured in years.

(a) What is the height of the tree five years after a seed is planted?

This is a separable differential equation and so first we separate to get

$$\frac{dH}{(40-H)^2} = \frac{1}{100}\,dt.$$

Now integrating we have

$$\frac{1}{40-H} = \frac{1}{100}t + C,$$

At time $t = 0$ the tree will be 0 feet tall (i.e., it hasn't started growing yet. Using the initial condition $H(0) = 0$ then we have

$$\frac{1}{40} = 0 + C \qquad \text{so that} \qquad \frac{1}{40-H} = \frac{1}{100}t + \frac{1}{40} = \frac{2t+5}{200}.$$

So we now have

$$40 - H = \frac{200}{2t+5} \qquad \text{or} \qquad H = 40 - \frac{200}{2t+5} = \frac{80t}{2t+5}.$$

Finally to answer the question we plug in $t = 5$ and see that after five years the tree will be $400/15 = 80/3$ feet or twenty-six feet and eight inches.

(b) What is the maximum height that the tree can grow to (i.e., what is the limit as $t \to \infty$ for the function H)?

Based on the answer to part (a) if we let $t \to \infty$ then $H \to 40$. Alternatively we see that 40 is an equilibrium solution and that for $H < 40$ we have $dH/dt > 0$ and so we will increase to the equilibrium solution $H = 40$.

7. Find $\int_0^{\pi/2} (\sin x + \cos x)^2 \sin x \, dx$.

We have

$$\begin{aligned}
\int_0^{\pi/2} (\sin x + \cos x)^2 \sin x \, dx &= \int_0^{\pi/2} (\sin^2 x + 2\sin x \cos x + \cos^2 x) \sin x \, dx \\
&= \int_0^{\pi/2} (1 + 2\sin x \cos x) \sin x \, dx \\
&= \int_0^{\pi/2} \sin x \, dx + 2 \underbrace{\int_0^{\pi/2} \sin^2 x \cos x \, dx}_{u = \sin x \qquad du = \cos x \, dx} \\
&= -\cos x \Big|_0^{\pi/2} + 2 \int_0^1 u^2 \, du \\
&= (0 + 1) + \frac{2}{3} u^3 \Big|_0^1 \\
&= 1 + \frac{2}{3} \\
&= \frac{5}{3}.
\end{aligned}$$

8. Taylor series can be defined for multivariable functions. For instance for a function of two variables x and y the degree 2 Taylor polynomial approximation around the point $(0,0)$ is given by

$$P_2(x,y)=f(0,0)+\frac{\partial f(0,0)}{\partial x}x+\frac{\partial f(0,0)}{\partial y}y+\frac{1}{2}\frac{\partial^2 f(0,0)}{\partial x^2}x^2+\frac{\partial^2 f(0,0)}{\partial x\partial y}xy+\frac{1}{2}\frac{\partial^2 f(0,0)}{\partial y^2}y^2.$$

(a) Find the degree 2 Taylor polynomial approximation for the function $f(x,y) = e^x\sqrt{y+9}$ around the point $(0,0)$.

This problem is not so much about Taylor polynomials as it is about finding the partial derivatives, evaluating them at $(0,0)$ and putting them into the right slot.

$$\begin{aligned}
f(x,y) &= e^x(y+9)^{1/2} & f(0,0) &= 3\\
\tfrac{\partial f(x,y)}{\partial x} &= e^x(y+9)^{1/2} & \tfrac{\partial f(0,0)}{\partial x} &= 3\\
\tfrac{\partial f(x,y)}{\partial y} &= \tfrac{1}{2}e^x(y+9)^{-1/2} & \tfrac{\partial f(0,0)}{\partial x} &= \tfrac{1}{6}\\
\tfrac{\partial^2 f(x,y)}{\partial x^2} &= e^x(y+9)^{1/2} & \tfrac{\partial^2 f(0,0)}{\partial x^2} &= 3\\
\tfrac{\partial^2 f(x,y)}{\partial x\partial y} &= \tfrac{1}{2}e^x(y+9)^{-1/2} & \tfrac{\partial^2 f(0,0)}{\partial x\partial y} &= \tfrac{1}{6}\\
\tfrac{\partial^2 f(x,y)}{\partial y^2} &= -\tfrac{1}{4}e^x(y+9)^{-3/2} & \tfrac{\partial^2 f(0,0)}{\partial x^2} &= -\tfrac{1}{108}
\end{aligned}$$

Putting these in we have that the Taylor polynomial is

$$P_2(x,y) = 3+3x+\frac{1}{6}y+\frac{3}{2}x^2+\frac{1}{6}xy-\frac{1}{216}y^2.$$

(b) Use the answer in part (a) to find an approximation for $\dfrac{\sqrt{10}}{e}$

Looking at the function we want to get $f(x,y) = e^x\sqrt{y+9} = \sqrt{10}/e$. We can do this by choosing $x=-1$ and $y=1$. So we have

$$\frac{\sqrt{10}}{e} = f(-1,1) \approx P_2(-1,1) = 3-3+\frac{1}{6}+\frac{3}{2}-\frac{1}{6}-\frac{1}{216} = \frac{323}{216}.$$

9. A ball is placed on a surface and released. If the surface is described by $z = 3x^2 + xy - 2y^2$ and the initial placement corresponds to the point above $x = 2$ and $y = 1$, find a unit vector (in the plane) which points in the direction that the ball will roll.

The ball will roll downhill in the direction corresponding to the steepest downhill slope. So we need to find a unit vector for which the directional derivative is as negative as possible. We know that the gradient points in the direction of greatest increase and opposite the gradient points in the direction of greatest decrease. So a vector (not necessarily a unit vector) is found by the negative of the gradient vector which is

$$-\begin{bmatrix} 6x + y \\ x - 4y \end{bmatrix}\Bigg|_{(x,y)=(2,1)} = \begin{bmatrix} -13 \\ 2 \end{bmatrix}.$$

We now need to make it a unit vector, first we find the magnitude which is $\sqrt{(-13)^2 + (2)^2} = \sqrt{173}$. Finally, we make it into a unit vector by dividing by it's length so that our final answer is

$$\begin{bmatrix} \dfrac{-13}{\sqrt{173}} \\ \dfrac{2}{\sqrt{173}} \end{bmatrix}.$$

10. Consider the function $h(x,y) = y^2 \sin x - x$.

(a) Verify that $\nabla h(0,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We have

$$\nabla h(x,y) = \begin{bmatrix} y^2 \cos x - 1 \\ 2y \sin x \end{bmatrix} \qquad \text{so} \qquad \nabla h(0,1) = \begin{bmatrix} 1^2 \cdot \cos 0 - 1 \\ 2 \cdot 1 \cdot \sin 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(b) Determine if the point $(0,1)$ is a maximum, minimum or saddle.

We have

$$\begin{aligned} D(x,y) &= h_{xx}(x,y) h_{yy}(x,y) - (h_{xy}(x,y))^2 \\ &= (-y^2 \sin x)(2 \sin x) - (2y \cos x)^2 \\ &= -2y^2 \sin^2 x - 4y^2 \cos^2 x. \end{aligned}$$

So we have $D(0,1) = -4$, since this value is negative this point corresponds to a saddle point.

11. Find the maximum and minimum values for $\kappa(x, y) = 3x - y$ for points on the curve $x^2 + 4y^2 = 10$.

This problem consists of finding extreme values given a constraint. In particular we will use the method of Lagrange multipliers to solve this problem. So we first introduce an auxiliary function

$$F(x, y, \lambda) = 3x - y - \lambda(x^2 + 4y^2 - 10).$$

We need to find the critical points of this function, and so we have

$$\begin{aligned}
\frac{\partial F(x, y, \lambda)}{\partial x} &= 3 - 2\lambda x &= 0 \\
\frac{\partial F(x, y, \lambda)}{\partial y} &= -1 - 8\lambda y &= 0 \\
\frac{\partial F(x, y, \lambda)}{\partial \lambda} &= x^2 + 4y^2 - 10 &= 0
\end{aligned}$$

From the first equation we have $x = 3/2\lambda$ and from the second we have $y = -1/8\lambda$. Putting these into the last equation we have

$$\frac{9}{4\lambda^2} + 4\frac{1}{64\lambda^2} = 10 \qquad \text{or (simplifying)} \qquad \lambda^2 = \frac{37}{160} \quad \text{so} \quad \lambda = \pm\sqrt{\frac{37}{160}}.$$

We now have that the critical points are at

$$(x_1, y_1) = \left(\frac{6\sqrt{370}}{37}, -\frac{\sqrt{370}}{74}\right) \quad \text{and} \quad (x_2, y_2) = \left(-\frac{6\sqrt{370}}{37}, \frac{\sqrt{370}}{74}\right).$$

Putting these in we have

$$\kappa(x_1, y_1) = \frac{1}{2}\sqrt{370} \quad \text{and} \quad \kappa(x_2, y_2) = -\frac{1}{2}\sqrt{370},$$

so that the maximum is $\sqrt{370}/2$ while the minimum is $-\sqrt{370}/2$.