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MATH 3B (Butler)

Midterm II, 20 February 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 12 points.

1. (a) Find the second degree ($n = 2$) Taylor polynomial for $f(x) = \sqrt[3]{x+8}$ around the point $x = 0$.

We have

$$\begin{aligned} f(x) &= (x+8)^{1/3} & \text{so} \quad f(0) &= 8^{1/3} = 2, \\ f'(x) &= \tfrac{1}{3}(x+8)^{-2/3} & \text{so} \quad f'(0) &= \tfrac{1}{12}, \\ f''(x) &= -\tfrac{2}{9}(x+8)^{-5/3} & \text{so} \quad f''(0) &= -\tfrac{1}{144}. \end{aligned}$$

So the second degree Taylor polynomial is

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 2 + \frac{1}{12}x - \frac{1}{288}x^2.$$

(b) Using the answer in part (a) give an approximation for $\sqrt[3]{7}$.

First we note that $\sqrt[3]{7} = f(-1)$. So putting -1 into the polynomial we found in part (a) we have

$$\sqrt[3]{7} \approx P_2(-1) = 2 - \frac{1}{12} - \frac{1}{288} = \frac{551}{288}.$$

(On a side note $\sqrt[3]{7} = 1.91293118\ldots$ while $\frac{551}{288} = 1.91319444\ldots$ and so the approximation is pretty good.)

2. Find $\displaystyle\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\,dx$.

This is an improper integral. Before we tackle that problem let us first find the indefinite integral (saving us some headache in notation). This can be done by noting that we can rewrite the integrals in terms of e^x and then make a substitution.

$$\underbrace{\int \frac{e^x}{e^{2x}+1}\,dx}_{\substack{u=e^x \\ du = e^x\,dx}} = \int \frac{1}{u^2+1}\,du = \arctan(u) + C = \arctan(e^x) + C$$

So we are now ready to deal with the improper integral

$$\begin{aligned}
\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\,dx &= \lim_{t\to-\infty} \int_t^0 \frac{e^x}{e^{2x}+1}\,dx \\
&= \lim_{t\to-\infty} \left(\arctan(e^x)\Big|_t^0 \right) \\
&= \lim_{t\to-\infty} \left(\arctan(e^0) - \arctan(e^t) \right) \\
&= \arctan(1) - \lim_{t\to-\infty} \left(\arctan(e^t) \right) \\
&= \frac{\pi}{4} - \arctan(0) \\
&= \frac{\pi}{4}.
\end{aligned}$$

3. It has recently been revealed that the reason that twinkies have such an incredible shelf life is due to a rare molecule known as Twinkonium (T). Each twinkie contains 5 grams of Twinkonium when first produced, unfortunately the Twinkonium then starts to decay and turn into ordinary sugar, through extensive research it has been determined that the rate of decay satisfies

$$\frac{dT}{dt} = -\frac{1}{10}T^2,$$

where t is time measured in years. If a twinkie is still good when it has at least 1 gram of Twinkonium, how many years from when it was first produced will the twinkie go bad?

We have that $T(0) = 5$ and we are trying to find a t so that $T(t) = 1$ given that

$$\frac{dT}{dt} = -\frac{1}{10}T^2.$$

Solving this separable equation we first have

$$\frac{dT}{T^2} = -\frac{1}{10}\,dt \quad \text{so} \quad \int \frac{1}{T^2}\,dT = \int -\frac{1}{10}\,dt \quad \text{or} \quad -\frac{1}{T} = -\frac{1}{10}t + C.$$

We could solve for T at this point, but we are not asked to and that would be extra work. First we note that the initial condition can be used to solve for C, in particular we have

$$-\frac{1}{5} = -\frac{1}{10}0 + C \quad \text{so} \quad C = -\frac{1}{5}.$$

Finally we can put in this value for C, set $T = 1$ and solve for t to get our final answer. So we have

$$-\frac{1}{1} = -\frac{1}{10}t - \frac{1}{5} \quad \text{or} \quad \frac{1}{10}t = \frac{4}{5} \quad \text{or} \quad t = 8.$$

So the twinkie will go bad in eight years.

(According to Hostess the twinkie has an "official" shelf life of 25 days.)

4. (a) Find $\int \frac{\ln x}{x}\,dx$.

This integral can easily be done using substitution. We have

$$\underbrace{\int \frac{\ln x}{x}\,dx}_{\substack{u=\ln x \\ du=\frac{1}{x}\,dx}} = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

(b) Find $\int x^a \ln x\,dx$, for $a \neq -1$.

This integral can be done using integration by parts. We have

$$\begin{aligned}
\underbrace{\int x^a \ln x\,dx}_{\substack{u=\ln x \qquad dv = x^a \\ du=\frac{1}{x}\,dx \qquad v=\frac{1}{a+1}x^{a+1}}} &= \frac{1}{a+1}x^{a+1}\ln x - \int \frac{1}{a+1}x^{a+1}\frac{1}{x}\,dx \\
&= \frac{1}{a+1}x^{a+1}\ln x - \frac{1}{a+1}\int x^a\,dx \\
&= \frac{1}{a+1}x^{a+1}\ln x - \frac{1}{(a+1)^2}x^{a+1} + C.
\end{aligned}$$

5. Find $\int \frac{1}{x^{3/2} - x^{1/2}}\,dx$. (Hint: $x^{3/2} = (x^{1/2})^3$.)

Using the hint we can rewrite this integral as

$$\int \frac{1}{(x^{1/2})^3 - x^{1/2}}\,dx.$$

This suggests that we try the substitution $u = x^{1/2}$ and so $du = \frac{1}{2}x^{-1/2}\,dx$, which can be rewritten as $dx = 2x^{1/2}\,du = 2u\,du$. So we have

$$\int \frac{1}{(x^{1/2})^3 - x^{1/2}}\,dx = \int \frac{2u}{u^3 - u}\,du = \int \frac{2}{u^2 - 1}\,du = \int \frac{2}{(u-1)(u+1)}\,du.$$

We recognize this last integral as a partial fractions problem. So first we want to break it up into pieces, i.e., find A and B so that

$$\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}.$$

Clearing the denominator on both sides we have

$$2 = A(u+1) + B(u-1).$$

We can now group coefficients (giving the two equations $A + B = 0$ and $A - B = 2$) or we can choose "nice" values for u. In the latter case choosing $u = 1$ we can conclude $A = 1$ and choosing $u = -1$ we can conclude $B = -1$. And so we have

$$\begin{aligned}\int \frac{2}{(u-1)(u+1)}\,du &= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\ &= \ln|u-1| - \ln|u+1| + C \\ &= \ln\left|\frac{u-1}{u+1}\right| + C.\end{aligned}$$

Finally we put it back in terms of x to get our final answer,

$$\int \frac{1}{x^{3/2} - x^{1/2}}\,dx = \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C.$$