

Midterm 2 Review

Substitution rule

Rules for derivatives easily transform into rules for integration. The most important rule for derivatives is the *chain rule*, i.e.,

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$

For integration this becomes the *substitution rule*

$$\int f(g(x))g'(x) dx = \int f(u) du, \quad \text{where } u = g(x).$$

This is used in many problems involving integration because it can help rewrite the integral in a simpler form. So after the substitution we might see how to proceed and then we can solve the integral and at the end *resubstitute* back to get our answer in terms of x . One indication that we should use substitution is that we look for a function inside of a function.

If we are dealing with a definite integral we can do one of two things. First, we do the indefinite integral, solve it to the end to get an antiderivative and then use the fundamental theorem of calculus to evaluate and get our answer. Alternatively, we can change the bounds as we make our substitution (the principle is that we are replacing *every* occurrence of x), i.e.,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du, \quad \text{where } u = g(x).$$

Integration by parts

Another rule for derivatives which we can transform into a rule for integration is the *product rule*, i.e.,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

For integration this becomes *integration by parts*

$$\int u dv = uv - \int v du.$$

There are a few things to look for that indicate when to use integration by parts. One obvious thing is if we see two functions multiplying together. Another one is when we see a function which we don't know how to integrate but has a simple derivative.

Our goal is always to simplify the integral so if doing integration by parts is making it worse then we should try to switch the roles of u and dv . Also, before we try integration by parts we should always first check for substitution (which is easier). Sometimes we must use a combination of substitution and integration by parts. Finally, some problems require using integration by parts several times.

If we are dealing with a definite integral we can again do one of two things. First, either do the indefinite integral to get an antiderivative and then use the fundamental theorem of calculus. Alternatively, we keep track of the bounds as we do the problem, i.e.,

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Rational functions

A rational function is one of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. We have already encountered several of these types of functions for integrating. For example we have the following:

$$\begin{aligned} \int u^k du &= \frac{1}{k+1}u^{k+1} + C, \quad k \neq -1 \\ \int \frac{1}{u} du &= \ln|u| + C \\ \int \frac{u}{u^2 + a^2} du &= \frac{1}{2} \ln(u^2 + a^2) + C \\ \int \frac{u}{(u^2 + a^2)^k} du &= \frac{1}{2(1-k)(u^2 + a^2)^{k-1}} + C \\ \int \frac{1}{u^2 + a^2} du &= \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\ \int \frac{1}{(u^2 + a^2)^k} du &= \left(\text{something which} \right. \\ &\quad \left. \text{we didn't cover} \right) \end{aligned}$$

Every integral involving a rational function can be rewritten into integrals of these types using tools from algebra. (Given a polynomial $P(x)$ the degree of $P(x)$ is the largest exponent in the polynomial.)

Steps to solve $\int \frac{P(x)}{Q(x)} dx$:

1. If $\text{degree}(P(x)) \geq \text{degree}(Q(x))$ then use long division to divide $Q(x)$ into $P(x)$. This allows us to rewrite
2. Factor $Q(x)$ into linear and *irreducible* quadratic terms (irreducible means that we cannot factor it further, i.e., $x^2 + 1$ is irreducible but $x^2 - 1$ is not).
3. Use partial fractions to break $\frac{P(x)}{Q(x)}$ into smaller parts.

If $Q(x) = (ax + b)^k \cdots$ then

$$\frac{P(x)}{Q(x)} = \frac{C_1}{ax + b} + \frac{C_2}{(ax + b)^2} + \cdots + \frac{C_k}{(ax + b)^k} + \cdots$$

If $Q(x) = (ax^2 + bx + c)^\ell$ then

$$\frac{P(x)}{Q(x)} = \frac{D_1x + E_1}{ax^2 + bx + c} + \cdots + \frac{D_\ell x + E_\ell}{(ax^2 + bx + c)^\ell} + \cdots$$

We now need to solve for the constants that were introduced. (As a rule of thumb the number of constants should be equal to the degree of $Q(x)$, if not then there is an error.) As a first step to do this clear the denominators on both sides (i.e., multiply through by $Q(x)$). There are then two approaches (which can also be combined). Either we (1) expand out and group coefficients, then since the polynomials on the left and right of the

equal sign are equal they must also have the same coefficients, this gives us a system of linear equations to solve; or we (2) choose “nice” values of x , i.e., values so that all but one term drops out and then we can read off the constants.

- Now integrate each individual term. Finishing the problem.

Improper integrals

An improper integral is one that involves infinity. This can happen in one of two ways, either the bounds involve ∞ , i.e., $\int_0^\infty e^{-x} dx$, or there is a vertical asymptote in the interval that we are integrating over, i.e., $\int_0^1 (x-1)^{-2/3} dx$. The method to deal with them is the same in both cases. We first approximate by an integral that does not involve ∞ and then take a limit of the approximation, i.e.,

$$\int_0^\infty e^{-x} dx = \lim_{t \rightarrow \infty} \left(\int_0^t e^{-x} dx \right) \quad \text{and}$$

$$\int_0^1 (x-1)^{-2/3} dx = \lim_{t \rightarrow 1^-} \left(\int_0^t (x-1)^{-2/3} dx \right).$$

An improper integral *converges* if it goes to a fixed (finite) number, otherwise it *diverges*. It is sometimes necessary to break an improper integral into smaller pieces and deal with each piece separately. This can happen if we are integrating from $-\infty$ to ∞ or if we have a vertical asymptote in the interior of the region we are integrating over (i.e., $\int_{-\infty}^\infty \frac{1}{x^2+1} dx$ or $\int_{-1}^1 \frac{1}{x^2} dx$). In this case if any one of the pieces diverges then the whole integral diverges.

Taylor polynomials

The idea behind Taylor polynomials is to find the best polynomial of degree n that approximates $f(x)$ near the point $x=a$ (tangent lines correspond to $n=1$). This polynomial is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Where $k! = k(k-1) \cdots 2 \cdot 1$ is “ k factorial” and $0! = 1$. We can use these polynomials to find numerical approximations, i.e., $f(x) \approx P_n(x)$ for x “near” a .

Solving differential equations

A differential equation is an equation involving the derivative. We will limit ourselves to separable first order differential equations, i.e., equations of the form

$$\frac{dy}{dx} = f(x)g(y) \quad \text{with} \quad y(x_0) = y_0.$$

Here “ $y(x_0) = y_0$ ” is an initial condition, i.e., we are looking for the function y which both satisfies the differential equation and the initial condition. When we solve these we will use integration which gives us a

constant C , this constant is then set so that the initial condition is satisfied. (We can solve for the constant C anytime after we have done the integration.)

Solving a separable differential equation:

- Separate. (Put all of the terms involving one variable on one side and all of the terms involving the other variable on the other side.)
- Integrate. (Take the indefinite integral of both sides, don’t forget to add a “ $+C$ ” on one side.)
- Simplify-erate. (Solve for C using the initial conditions and also solve for y .)

Once we have solved the differential equation we can then use the function y to answer various questions, i.e., at what value x does y hit a certain value, or what value is y at a time x , and so on.

(There is nothing special about x and y , we can use any symbols that we like, the process is unchanged. Sometimes a differential equation does not look separable at first glance, but with some rewriting we can put it into the form of a separable equation.)

We might need to set up a differential equation before we solve it. In that case it is useful to remember the general rubric:

$$\left(\begin{array}{l} \text{rate at which} \\ y \text{ is changing} \end{array} \right)$$

$$= \left(\begin{array}{l} \text{rate at which} \\ y \text{ is increasing} \end{array} \right) - \left(\begin{array}{l} \text{rate at which} \\ y \text{ is decreasing} \end{array} \right).$$

So identify what is causing y to increase or decrease, compute the appropriate rates and then combine.

Stability of equilibria

For a differential equation of the form $\frac{dy}{dx} = g(y)$ (also called autonomous), an equilibrium solution is a constant solution, i.e., $y = \hat{y}$ (where \hat{y} is a constant). These are easily found by solving $g(y) = 0$.

An equilibrium solution \hat{y} can be either *stable* or *unstable*. A stable solution is one where values of y near \hat{y} head towards \hat{y} , while an unstable solution is one where values of y near \hat{y} move away from \hat{y} .

To determine if an equilibrium solution is stable or unstable we can graph $g(y)$ as a function of y (the points \hat{y} correspond to where the curve intersects the horizontal axis), where it is positive corresponds to where y would be increasing and where it is negative corresponds to where y is decreasing (this can be marked using arrows on the horizontal axis). A stable point is one where the arrows point *towards* the point while an unstable point is one where the arrows point *away* from the point.

Alternatively we can skip drawing and observe that stability depends on what is happening “locally” and so we only need to know how the graph crosses the axis (i.e., uphill or downhill). To figure this out we compute $g'(\hat{y})$ (also called the eigenvalue), if $g'(\hat{y}) < 0$ then the solution is stable, if $g'(\hat{y}) > 0$ then the solution is unstable, finally if $g'(\hat{y}) = 0$ then we don’t know and should draw the picture.