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MATH 3B (Butler)

Midterm II, 20 February 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 12 points.

1. (a) Find the second degree ($n = 2$) Taylor polynomial for $f(x) = \sqrt{x+9}$ around the point $x = 0$.

We have

$$\begin{aligned} f(x) &= (x+9)^{1/2} &\quad \text{so} \quad f(0) &= 9^{1/2} = 3, \\ f'(x) &= \tfrac{1}{2}(x+9)^{-1/2} &\quad \text{so} \quad f'(0) &= \tfrac{1}{6}, \\ f''(x) &= -\tfrac{1}{4}(x+9)^{-3/2} &\quad \text{so} \quad f''(0) &= -\tfrac{1}{108}. \end{aligned}$$

So the second degree Taylor polynomial is

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 3 + \frac{1}{6}x - \frac{1}{216}x^2.$$

(b) Using the answer in part (a) give an approximation for $\sqrt{10}$.

First we note that $\sqrt{10} = f(1)$. So putting 1 into the polynomial we found in part (a) we have

$$\sqrt{10} \approx P_2(1) = 3 + \frac{1}{6} - \frac{1}{216} = \frac{683}{216}.$$

(On a side note $\sqrt{10} = 3.16227766\ldots$ while $\frac{683}{216} = 3.16203704\ldots$ and so the approximation is pretty good.)

2. Find $\displaystyle\int_0^e \frac{1}{x(1+(\ln x)^2)}\,dx.$

This is an improper integral (at 0 the denominator is 0, though that is not completely obvious). Before we tackle that problem let us first find the indefinite integral (saving us some headache in notation). This can be done by making a substitution.

$$\underbrace{\int \frac{1}{x(1+(\ln x)^2)}\,dx}_{\substack{u=\ln x\\ du=\frac{1}{x}\,dx}} = \int \frac{1}{u^2+1}\,du = \arctan(u)+C = \arctan(\ln x)+C$$

So we are now ready to deal with the improper integral

$$\begin{aligned}
\int_0^e \frac{1}{x(1+(\ln x)^2)}\,dx &= \lim_{t\to 0^+}\int_t^e \frac{1}{x(1+(\ln x)^2)}\,dx \\
&= \lim_{t\to 0^+}\left(\arctan(\ln x)\Big|_t^e\right) \\
&= \lim_{t\to 0^+}\left(\arctan(\ln e)-\arctan(\ln t)\right) \\
&= \arctan(1)-\lim_{t\to 0^+}\left(\arctan(\ln t)\right) \\
&= \frac{\pi}{4}-\left(-\frac{\pi}{2}\right) \\
&= \frac{3\pi}{4}.
\end{aligned}$$

At the end we used that $\ln t \to -\infty$ and that the horizontal asymptote of $\arctan x$ is $-\pi/2$ as $x\to-\infty$.

3. After a recent storm a dividing wall that separated the sea from a small pond has cracked and salt water is now flowing into the pond. The pond initially held 10,000 gallons of fresh water (i.e., no salt). After the break in the wall the sea water started coming in at a rate of 100 gallons per minute, and simultaneously water left the pond at a rate of 100 gallons per minute in an overflow channel. If Q is the amount of salt in ounces and we assume that the seawater has 4 ounces of salt in every gallon then we can model the rate of change of the amount of salt in the pond by the following differential equation.

$$\frac{dQ}{dt} = 400 - \frac{1}{100}Q.$$

(a) Solve the differential equation for Q as a function of time (where $t = 0$ corresponds to the time when the wall broke).

We have that $Q(0) = 0$ (i.e., no salt initially when the wall first breaks. Solving this separable equation we have

$$\frac{dQ}{dt} = -\frac{1}{100}(Q - 40000) \quad \text{or} \quad \frac{dQ}{Q - 40000} = -\frac{1}{100}dt.$$

Integrating both sides we get

$$\ln|Q - 40000| = -\frac{1}{100}t + C \quad \text{or} \quad Q - 40000 = e^{-t/100+C} = De^{-t/100}.$$

Using the initial condition that $Q(0) = 40000$ we have that $0 - 40000 = De^0 = D$, substituting this in and solving for Q we can conclude that

$$Q = 40000 - 40000e^{-t/100}.$$

(b) What is the amount of salt in the pond after a long time has elapsed (i.e., what is $\lim_{t\to\infty} Q(t)$)? (Hint: you do not need to know the answer for part (a) to get the answer to part (b).)

As $t \to \infty$ the term $e^{-t/100} \to 0$ and so in the long run Q goes to 40000. This can also be seen in that in the long run all of the water in the pond will become seawater which has 4 ounces per gallon, and since there are 10000 gallons we will have 40000 ounces of salt.

4. Find $\int t^2 e^{-t/2}\,dt$.

This integral can be done using repeated integration by parts. We have

$$\begin{aligned}
\underbrace{\int t^2 e^{-t/2}\,dt}_{\substack{u = t^2 \qquad dv = e^{-t/2} \\ du = 2t\,dt \qquad v = -2e^{-t/2}}} &= t^2(-2e^{-t/2}) + 4\Bigg(\underbrace{\int t e^{-t/2}\,dt}_{\substack{u = t \qquad dv = e^{-t/2} \\ du = dt \qquad v = -2e^{-t/2}}} \Bigg) \\
&= -2t^2 e^{-t/2} + 4\left(-2te^{-t/2} + 2\int e^{-t/2}\,dt \right) \\
&= -2t^2 e^{-t/2} - 8te^{-t/2} - 16e^{-t/2} + C.
\end{aligned}$$

5. Find $\int \frac{e^x}{e^{2x}-1}\,dx$. (Hint: $e^{2x} = (e^x)^2$.)

The hint suggests that we try the substitution $u = e^x$ and so $du = e^x\,dx$. So we have

$$\int \frac{e^x}{e^{2x}-1}\,dx = \int \frac{1}{u^2-1}\,du = \int \frac{1}{(u-1)(u+1)}\,du$$

We recognize this last integral as a partial fractions problem. So first we want to break it up into pieces, i.e., find A and B so that

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}.$$

Clearing the denominator on both sides we have

$$1 = A(u+1) + B(u-1).$$

We can now group coefficients (giving the two equations $A + B = 0$ and $A - B = 1$) or we can choose "nice" values for u. In the latter case choosing $u = 1$ we can conclude $A = 1/2$ and choosing $u = -1$ we can conclude $B = -1/2$. And so we have

$$\begin{aligned}\int \frac{1}{(u-1)(u+1)}\,du &= \int \left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right)du \\ &= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C \\ &= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \;=\; \ln\sqrt{\left|\frac{u-1}{u+1}\right|} + C.\end{aligned}$$

Finally we put it back in terms of x to get our final answer,

$$\int \frac{e^x}{e^{2x}-1}\,dx = \ln\sqrt{\left|\frac{e^x-1}{e^x+1}\right|} + C.$$