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MATH 3B (Butler)
Midterm I, 26 January 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 10 points.

1. Find

$$\int (\sec u + \tan u)^2 du.$$

(Hint: it might be helpful to know that $\sec^2 u = \tan^2 u + 1$.)

We have

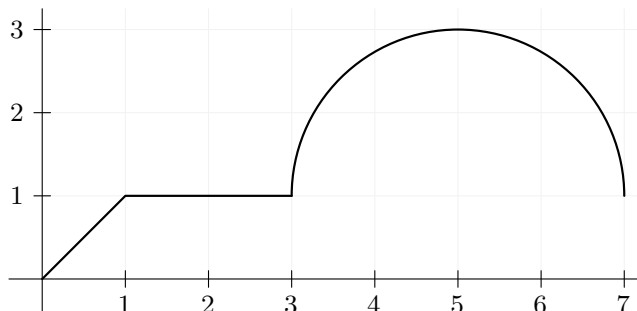
$$\begin{aligned} \int (\sec u + \tan u)^2 du &= \int (\sec^2 u + 2 \sec u \tan u + \tan^2 u) du \\ &= \int (\sec^2 u + 2 \sec u \tan u + (\sec^2 u - 1)) du \\ &= \int (2 \sec^2 u + 2 \sec u \tan u - 1) du \\ &= 2 \tan u + 2 \sec u - u + C. \end{aligned}$$

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2. Let $f(x)$ be the function defined piecewise on the interval $[0, 7]$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } 1 \leq x \leq 3, \\ 1 + \sqrt{4 - (x - 5)^2} & \text{if } 3 \leq x \leq 7. \end{cases}$$

A picture of this function is shown below.



Given that $F(x) = \int_2^x f(t) dt$ find the following.

(a) $F(0) =$

$$F(0) = \int_2^0 f(t) dt = - \int_0^2 f(t) dt.$$

This last integral consists of a triangle with base and height 1 (so area $1/2$) and a square with base and height 1 (so area 1). Combining we can conclude that $F(0) = -3/2$.

(b) $F(2) =$

$$F(2) = \int_2^2 f(t) dt = 0.$$

(This is one of our basic properties of integrals.)

(c) $F(7) =$

$$F(7) = \int_2^7 f(t) dt.$$

This region can be broken into two parts, a rectangle with height 1 and width 5 (so area of 5) and half of a circle of radius 2 (so area $(1/2)\pi(2)^2 = 2\pi$). Combining we can conclude that $F(7) = 5 + 2\pi$.

3. Let $h(x)$ be a function such that

$$\int_0^2 h(x) dx = 1, \int_0^3 h(x) dx = 2, \int_0^4 h(x) dx = 6, \int_1^5 h(x) dx = 5, \text{ and } \int_2^5 h(x) dx = 7.$$

Find $\int_1^3 h(x) dx$.

Our basic approach is to combine these integrals for which we have a value in various ways to get information about the integral we are interested in. One approach is as follows:

$$\int_0^5 h(x) dx = \int_0^2 h(x) dx + \int_2^5 h(x) dx = 1 + 7 = 8.$$

Using this we have

$$\int_0^1 h(x) dx = \int_0^5 h(x) dx - \int_1^5 h(x) dx = 8 - 5 = 3,$$

So finally we have

$$\int_1^3 h(x) dx = \int_0^3 h(x) dx - \int_0^1 h(x) dx = 2 - 3 = -1.$$

(Note that we did not need to use all of the information given to us.)

4. For $x > 0$ let $G(x) = \int_{\ln x}^{2 \ln x} e^{2v} dv$. Find the critical value of $G(x)$.

To find the critical value we need to solve $G'(x) = 0$. So using Leibniz's rule we have

$$\begin{aligned} G'(x) &= \frac{d}{dx} \left(\int_{\ln x}^{2 \ln x} e^{2v} dv \right) \\ &= e^{4 \ln x} \frac{2}{x} - e^{2 \ln x} \frac{1}{x} \\ &= e^{\ln x^4} \frac{2}{x} - e^{\ln x^2} \frac{1}{x} \\ &= x^4 \frac{2}{x} - x^2 \frac{1}{x} \\ &= 2x^3 - x. \end{aligned}$$

We now need to solve $2x^3 - x = (2x^2 - 1)x = 0$. Since $x > 0$ this leaves us with only one possibility, $x = \sqrt{1/2} = \sqrt{2}/2$.

5. Find the area in the region between the curves $g(t) = \sin(\pi t)$ and $f(t) = 8t^2 - 2t$. (Hint: the intersection of these curves occur for simple values of t .)

Since we are not given bounds we first need to figure out the bounds. This reduces to solving for t in the following equation.

$$\sin(\pi t) = 8t^2 - 2t$$

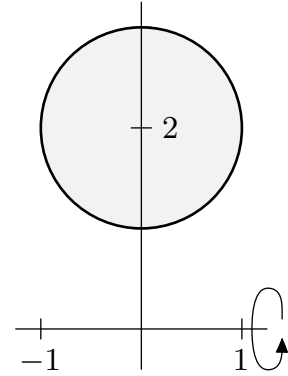
There is *no* method to solve this analytically, and since no calculators are allowed then it must be that we can guess solutions. For example $t = 0$ gives zero on both sides and so that is a point of intersection. The next “nice” value for $\sin(\pi t)$ would occur at $t = 1/2$ and if we plug that in we see that we get 1 on both sides. It is not hard to see that these are the only two points of intersection (if in doubt do a rough sketch of these two functions). Now let us try to see which function is on top, if we pick $t = 1/4$ (a point in the middle) we see that $g(1/4) = \sin(\pi/4) = \sqrt{2}/2$ while $f(1/4) = 8(1/4)^2 - 2(1/4) = 0$, so $g(t)$ is the function on the top and $f(t)$ is the function on the bottom (this last part is easy to see if you do a rough sketch of the functions).

So we have

$$\begin{aligned} \text{Area} &= \int_0^{1/2} (g(t) - f(t)) dt \\ &= \int_0^{1/2} (\sin(\pi t) - (8t^2 - 2t)) dt \\ &= \int_0^{1/2} (\sin(\pi t) - 8t^2 + 2t) dt \\ &= \left(-\frac{1}{\pi} \cos(\pi t) - \frac{8}{3}t^3 + t^2 \right) \Big|_0^{1/2} \\ &= \left(-\frac{1}{\pi} \cos\left(\frac{\pi}{2}\right) - \frac{8}{3}\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 \right) - \left(-\frac{1}{\pi} \cos(0) - 0 + 0 \right) \\ &= \frac{1}{\pi} - \frac{1}{12}. \end{aligned}$$

6. A torus (or as Homer Simpson would say “mmmmm, donut”) can be formed by spinning a circle around the x -axis. Find the volume of the torus found by spinning the circle of radius 1 centered at $(0, 2)$ around the x -axis.

(Hint: the curve describing the top of the circle is $y = 2 + \sqrt{1 - x^2}$ while the curve describing the bottom of the circle is $y = 2 - \sqrt{1 - x^2}$.)



This is finding a solid of revolution, for which we have some slick formulas for. In particular we have

$$\begin{aligned}
 \text{Volume} &= \pi \int_{-1}^1 \left((2 + \sqrt{1 - x^2})^2 - (2 - \sqrt{1 - x^2})^2 \right) dx \\
 &= \pi \int_{-1}^1 \left((4 + 4\sqrt{1 - x^2} + (1 - x^2)) - (4 - 4\sqrt{1 - x^2} + (1 - x^2)) \right) dx \\
 &= 8\pi \int_{-1}^1 \sqrt{1 - x^2} dx \\
 &= 8\pi(\pi/2) = 4\pi^2.
 \end{aligned}$$

For the last step we used the fact that $\int_{-1}^1 \sqrt{1 - x^2} dx$ is the area of half of a circle of radius 1, i.e., has value $\pi/2$.