Midterm 1 Review

Riemann sums

If we want to approximate an area we can slice it into little strips each of which can be approximated by a rectangle; we then add up the individual rectangles. To get a better approximation we can make the slices "smaller". This is the underlying idea of *Riemann sums*. Given a function $f(x)$ and an interval $[a,b]$ we start by partitioning $[a,b]$ up into a partition P into pieces by first choosing points

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These are the bases of the rectangle and the "width" of the kth rectangle is $\Delta x_k = x_k - x_{k-1}$. To find the heights we choose points c_k so that $x_{k-1} \le c_k \le x_k$ we then have that the "height" is $f(c_k)$. So then we have that the Riemann sum (which is an approximation of the area under the curve $y = f(x)$ in the interval $[a,b]$) is

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Here "Σ" is used to indicate doing a sum. To indicate the slices getting smaller we let $\|P\| = \max\{\Delta x_k\}$ and so we want $\|P\| \to 0$. We are interested in functions where the limit as $\|P\| \to 0$ exists, we call such functions integrable (all continuous functions are integrable) and denote the limit by

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx.$$

The $\int$ sign is a stretched out "S" and indicates the idea that we are summing up little pieces. The "x" is a dummy variable and can be replaced by any other variable, the result will be the same.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(y)\,dy.$$

While our starting point is thinking of finding area, it is important to remember that the result of the integration can be positive or negative, so more appropriately it is signed area.

While Riemann sums underly the principles of integration, in this test we will *not* test you directly on Riemann sums (including problems involving Σ).

Properties of integrals

Properties of integration follow from the definition of Riemann sums (as well as some geometric intuition).

$$\int_a^b f(x)\,dx = \begin{bmatrix}\text{area above}\\ x\text{-axis}\end{bmatrix} - \begin{bmatrix}\text{area below}\\ x\text{-axis}\end{bmatrix}.$$

We can find the values of some integrals by finding the area is composed of combinations of triangles, rectangles and circles.

If our upper and lower bound match then there is no "area" and so the integral is 0, i.e.,

$$\int_a^a f(x)\,dx = 0.$$

Changing the order of integration changes the sign, i.e.,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Integration is "linear" in the sense that we can pull constants out as well as break it up over addition, i.e.,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad \text{and}$$

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We can break the interval we are integrating into pieces (this is convenient, for example, when we have piecewise functions), i.e.,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This is true for any relationship of a, b, c.

If $f(x) \ge 0$ on $[a,b]$ then

$$\int_a^b f(x)\,dx \ge 0.$$

More generally, if $m \le f(x) \le M$ on $[a,b]$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the ideas of differentiation with our new idea of integration. There are two parts to the Fundamental Theorem of Calculus.

(Part I) If f is continuous on $[a,b]$ and

$$F(x) = \int_a^x f(u)\,du \quad \text{then} \quad F'(x) = f(x).$$

By combining this part of the Fundamental Theorem of Calculus, the chain rule and properties of integrals we have Leibniz's rule:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x).$$

We see from part I of the Fundamental theorem of calculus that the function $F(x)$ is an anti-derivative of $f(x)$. So we will let $\int f(x)\,dx$ (called the indefinite integral) denote the anti-derivative of $f(x)$. In general we have that

$$\int f(x)\,dx = C + \int_a^x f(x)\,dx,$$

where C is a constant (this constant will play an important role later, it is important not to forget it). Using what we know about derivatives we can now generate the following short list of indefinite integrals:

$$\begin{aligned}
\int x^k\,dx &= \frac{1}{k+1}x^{k+1} + C, \quad \text{if } k \neq -1\\
\int \frac{1}{x}\,dx &= \ln|x| + C\\
\int e^x\,dx &= e^x + C\\
\int \sin x\,dx &= -\cos x + C\\
\int \cos x\,dx &= \sin x + C\\
\int \sec^2 x\,dx &= \tan x + C\\
\int \sec x \tan x\,dx &= \sec x + C\\
\int \frac{1}{1+x^2}\,dx &= \arctan x + C
\end{aligned}$$

We can also use anti-derivatives to compute definite integrals.

(Part II) If f is continuous on $[a, b]$ then

$$\int_a^b f(u)\,du = F(b) - F(a),$$

where $F(x)$ is *any* anti-derivative of $f(x)$.

In particular if we want to evaluate a definite integral we can now do it in two steps. First, find an anti-derivative of the function. Second, evaluate this function at the endpoints and take the difference.

Applications of integration

Area: We can use integration to find the area between curves. If $g(x) \leq f(x)$ on the interval $[a, b]$ then the area between these curves in the interval is

$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

If the curves cross then find the intersection point(s) by setting $f(x) = g(x)$ and solving for x (also done when no bounds are given). Once we have the intersection point(s) we split this into several pieces and work on each piece separately (this is to avoid the problem of "signed" areas).

We can also integrate with respect to y, the basic idea being to take horizontal slices. In this case if we have $x = f(y)$ and $x = g(y)$ with $g(y) \leq f(y)$ on the interval $[a, b]$. Then we have

$$\text{Area} = \int_a^b \big(f(y) - g(y)\big)\,dy.$$

Volume: We can use integration to find volume. Intuitively this is done by slicing into cross sections and then adding up the cross sections. So we have that

$$\text{Volume} = \int_a^b A(x)\,dx,$$

where $A(x)$ is the area of the slice of the cross section. One special case is when we form a solid by revolving a region around the x-axis. If we revolve the region between $y = f(x)$ and the x-axis then a cross section is a circle and so the area of the cross section is $\pi\big(f(x)\big)^2$. So in this case we have

$$\text{Volume} = \pi \int_a^b \big(f(x)\big)^2\,dx.$$

If we want to spin a region between two curves $g(x)$ and $f(x)$ with $g(x) \leq f(x)$ then the resulting cross sections are annuli and the area of the cross sections are the difference of circles. Applying the same rule we have

$$\text{Volume} = \pi \int_a^b \Big(\big(f(x)\big)^2 - \big(g(x)\big)^2\Big)\,dx.$$

Cumulative change: We can use integration to add up the small instantaneous changes to get the total change.

$$\int_a^b \frac{dN}{dt}\,dt = N(b) - N(a) = \begin{pmatrix}\text{amount of change}\\ \text{betwen } t = a \text{ and } t = b\end{pmatrix}$$

For example, by integrating velocity (the derivative of position) we get the change of position.

Average value: The average value of f in the interval $[a, b]$ is given by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This value is such that the rectangle with height f_{avg} and width $(b - a)$ has the same area as $\int_a^b f(x)\,dx$.

The *Mean Value Theorem for Definite Integrals* states that if f is continuous on $[a, b]$ then there is some $c \in [a, b]$ so that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$