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MATH 3B (Butler)

Final, 20 March 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 10 points.

1. Find the area between the curves $y = x^3$ and $x = y^2$.

This is the same as finding the area between $y = x^3$ and $y = \sqrt{x}$. It is easy to see that these two curves intersect at $(0,0)$ and $(1,1)$ and that $\sqrt{x} > x^3$ for x between 0 and 1. So the area is

$$\int_0^1 \left(x^{1/2} - x^3\right) dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \left(\frac{2}{3} - \frac{1}{4}\right) - (0 - 0) = \frac{5}{12}.$$

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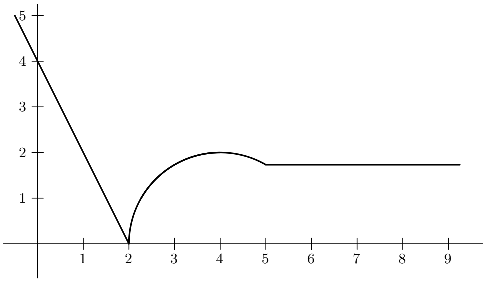
2. Let $H(x) = \int_{x^2}^{x+2} h(t)\,dt$ where $h(t)$ is the function defined piecewise by

$$h(t) = \begin{cases} 4-2t & \text{if } t \le 2, \\ \sqrt{4-(t-4)^2} & \text{if } 2 \le t \le 5, \\ \sqrt{3} & \text{if } t \ge 5. \end{cases}$$



A graph of this function is shown on the right. Find the following values.

(a) $H(0) =$

Using the area formula for a triangle, we have

$$H(0) = \int_0^2 h(t)\,dt = \frac{1}{2}\cdot 2\cdot 4 = 4.$$

(b) $H'(1) =$

Using Leibniz's rule we have that $H'(x) = h(x+2)\cdot 1 - h(x^2)\cdot 2x$, so we have

$$H'(1) = h(3) - 2h(1) = \sqrt{3} - 2\cdot 2 = \sqrt{3} - 4.$$

(c) $H(2) =$

Using basic properties of integration we have

$$H(2) = \int_4^4 h(t)\,dt = 0.$$

(d) $H(3) =$

Using the area formula for a rectangle, we have

$$H(3) = \int_9^5 h(t)\,dt = -\int_5^9 h(t)\,dt = -4\sqrt{3}.$$

3. Find $\displaystyle\int \frac{4\cos\theta}{3+\cos^2\theta}\,d\theta$.

Using the identity $\cos^2\theta + \sin^2\theta = 1$ we can rewrite this as

$$\int \frac{4\cos\theta}{3+\cos^2\theta}\,d\theta = \int \frac{4\cos\theta}{3+(1-\sin^2\theta)}\,d\theta = \int \frac{4\cos\theta}{4-\sin^2\theta}\,d\theta.$$

Now we make a substitution $u = \sin\theta$ with $du = \cos\theta\,d\theta$ so that the integral now becomes

$$\int \frac{4\cos\theta}{4-\sin^2\theta}\,d\theta = \int \frac{4\,du}{4-u^2} = \int \frac{4\,du}{(2+u)(2-u)}.$$

Once we get in this last form we see that this is a partial fractions problem. So we need to solve for A and B where

$$\frac{4}{4-u^2} = \frac{A}{2+u} + \frac{B}{2-u}.$$

Clearing denominators we have $4A = A(2-u) + B(2+u)$. Choosing $u = 2$ we see that $A = 1$ and choosing $u = -2$ we see that $B = 1$. So we now have

$$\begin{aligned}
\int \frac{4\,du}{(2+u)(2-u)} &= \int \left(\frac{1}{2+u} + \frac{1}{2-u}\right) du \\
&= \ln|2+u| - \ln|2-u| + C \\
&= \ln\left|\frac{2+u}{2-u}\right| + C \\
&= \ln\left|\frac{2+\sin\theta}{2-\sin\theta}\right| + C
\end{aligned}$$

4. Find $\int_0^\infty t^2 e^{-\lambda t}\,dt$ for $\lambda > 0$.

This is an improper integral. Before we handle that though let us first focus on doing the integral. This is a classic integration by parts problem, and for us we will start by letting $u = t^2$ so that the term becomes simpler. We then have to repeat this process a second time. So we have

$$\begin{aligned}
\underbrace{\int t^2 e^{-\lambda t}\,dt}_{\substack{u = t^2 \quad dv = e^{-\lambda t}\,dt \\ du = 2t\,dt \quad v = \frac{-1}{\lambda}e^{-\lambda t}}} &= -\frac{1}{\lambda}t^2 e^{-\lambda t} + \frac{2}{\lambda}\Bigg(\underbrace{\int t e^{-\lambda t}\,dt}_{\substack{u = t \quad dv = e^{-\lambda t}\,dt \\ du = dt \quad v = \frac{-1}{\lambda}e^{-\lambda t}}} \Bigg) \\
&= -\frac{1}{\lambda}t^2 e^{-\lambda t} + \frac{2}{\lambda}\Bigg(-\frac{1}{\lambda}te^{-\lambda t} + \frac{1}{\lambda}\int e^{-\lambda t}\,dt\Bigg) \\
&= \frac{-1}{\lambda}t^2 e^{-\lambda t} - \frac{2}{\lambda^2}te^{-\lambda t} - \frac{2}{\lambda^3}e^{-\lambda t} + C.
\end{aligned}$$

Now we are ready to deal with the improper integral. We have

$$\begin{aligned}
\int_0^\infty t^2 e^{-\lambda t}\,dt &= \lim_{x\to\infty}\int_0^x t^2 e^{-\lambda t}\,dt \\
&= \lim_{x\to\infty}\left(\frac{-\lambda^2x^2 - 2\lambda x - 2}{\lambda^3 e^{\lambda x}} + \frac{2}{\lambda^3}\right).
\end{aligned}$$

The first term is going toward ∞/∞ so we can use L'Hospital's rule to help us evaluate the limit. In particular, we have

$$\lim_{x\to\infty}\frac{-\lambda^2x^2 - 2\lambda x - 2}{\lambda^3 e^{\lambda x}} = \lim_{x\to\infty}\frac{-2\lambda^2 x - 2\lambda}{\lambda^4 e^{\lambda x}} = \lim_{x\to\infty}\frac{-2\lambda^2}{\lambda^5 e^{\lambda x}} = 0.$$

So we can conclude that

$$\int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^3}.$$

5. Differential equations can be used, among many other things, to model the spread of contagions through a population. For instance, in 1962 a young girl at a boarding school in a rural village of Tanganyika (now Tanzania) started laughing, her laughter quickly spread to her classmates then to her village and then to surrounding villages, this epidemic of laughter continued for several weeks. Suppose that we assume that we are more likely to start laughing in a laughing epidemic the longer we have been exposed to it and at the same time that it spreads fastest when there are fewer people currently laughing. We can roughly capture this in the differential equation

$$\frac{dY}{dt} = -\frac{1}{100}tY,$$

where Y represents the total number of people who are currently *not* laughing and t is time measured in hours.

(a) Suppose that initially there are 50000 people who are not laughing. Solve for Y as a function of t.

Using our technique to solve separable differential equations we have

$$\frac{dY}{Y} = -\frac{1}{100}t\,dt \quad \text{so integrating we have} \quad \ln|Y| = -\frac{1}{200}t^2 + C,$$

we can now exponentiate both sides (and move the constant out of the exponent to a new constant in front) giving

$$Y = De^{-t^2/200}.$$

Finally, we need to solve for D which is easily done using initial conditions since $Y(0) = 50000 = D$. So we have

$$Y = 50000e^{-t^2/200}.$$

(b) Using the answer in part (a), how many hours until there are only 25000 people not laughing? How many hours until there are only 10000 people not laughing?

To find the time it takes to get down to 25000 we solve for t in the following

$$25000 = 50000e^{-t^2/200} \quad \text{or} \quad \frac{t^2}{200} = \ln 2 \quad \text{or} \quad t = \sqrt{200\ln 2}.$$

Similarly, to find the time it takes to get to 10000 we have

$$10000 = 50000e^{-t^2/200} \quad \text{or} \quad \frac{t^2}{200} = \ln 5 \quad \text{or} \quad t = \sqrt{200\ln 5}.$$

6. Fill in the box so that the function $f(x,y)$ is continuous for all (x,y).

$$f(x,y)=\begin{cases} \dfrac{\displaystyle\int_x^y \cos(t^2)\,dt}{y-x} & \text{if } x\neq y,\\[2ex] \boxed{} & \text{if } x=y. \end{cases}$$

Briefly (in ten lines or less) explain your answer.

In order for the function to be continuous we need to determine what happens to the function as we let $x\to y$ (i.e., what should happen when $x=y$).

Method 1:
The function for when $x\neq y$ is the average value for the function $\cos(t^2)$ between x and y. So by the mean value theorem we have that $f(x,y)=\cos(c^2)$ for some c between x and y. Now to make the function continuous we need to have what happens at $x=y$ to be the same as what happens as we approach points along $x=y$ (i.e., take a limit). So as y gets closer to x then the value of c, which is "sandwiched" between x and y, approaches x so that the function approaches

$$\boxed{\cos(x^2)}$$

Method 2:
When $x=y$ we see that the top term will be 0 since our integral starts and stops at the same point while the bottom term is also 0. So we have 0/0, which is undefined. But this reminds us of L'Hospital's rule, and so we will fix x and take the limit as $y\to x$ to see what it "should" be. We have using the fundamental theorem of calculus that

$$\lim_{y\to x}\frac{\displaystyle\int_x^y \cos(t^2)\,dt}{y-x}=\lim_{y\to x}\frac{\cos(y^2)}{1}=\boxed{\cos(x^2)}$$

7. A parametric curve is used to describe how a point moves through space and is done by giving a function for each coordinate, namely, $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$. The tangent vector of the curve is found by taking the derivative of each entry, namely, $\mathbf{r}'(t) = \big[x'(t), y'(t), z'(t)\big]'$, and can be used to indicate the current direction in which the point is moving.

(a) Find $\mathbf{r}'(t)$ for $\mathbf{r}(t) = (4 - 3\arctan t, 5 + \ln(e^t + t), 7 - 2t)$.

This is testing to see if we can follow directions. The definition above tells us that to find the vector $\mathbf{r}'(t)$ we take the entry of each component. So we have

$$\mathbf{r}'(t) = \begin{bmatrix} \dfrac{-3}{1+t^2} \\ \dfrac{e^t + 1}{e^t + t} \\ -2 \end{bmatrix}.$$

(b) A plane is perpendicular to a parametric curve $\mathbf{r}(t)$ at t_0 if it contains the point $\mathbf{r}(t_0)$ and is perpendicular to $\mathbf{r}'(t_0)$. Give a formula for the plane perpendicular to the parametric curve in part (a) at $t = 0$.

To find a plane (which is what we are told to do) we need a point and a normal vector. The directions tell us that our point is the point on the curve

$$\mathbf{r}(0) = (4, 5, 7),$$

and that the normal vector (the vector perpendicular to the plane) can be found using $\mathbf{r}'(0)$ since by definition it will be perpendicular to the plane,

$$\mathbf{r}'(0) = \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}.$$

Now using the equation for the plane (conveniently given to us on the equation sheet) we have that the desired plane is

$$-3(x-4) + 2(y-5) - 2(z-7) = 0 \quad \text{or} \quad -3x + 2y - 2z + 16 = 0.$$

8. Find (x_0, y_0) so that the plane tangent to the surface $z = f(x, y) = x^2 + 3xy - y^2$ at $(x_0, y_0, (f(x_0, y_0))$ is *parallel* to the plane $16x - 2y - 2z = 23$.

We want two planes to be parallel, so we need that the normal vectors are scalar multiples of each other. The plane $16x - 2y - 2z = 23$ has $[16, -2, -2]$ as its normal vector while the tangent plane

$$z = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

has $\left[\frac{\partial f(x_0, y_0)}{\partial x}, \frac{\partial f(x_0, y_0)}{\partial y}, -1\right]$ as its normal vector. So for some value of α we need

$$\begin{bmatrix} \frac{\partial f(x_0, y_0)}{\partial x} \\ \frac{\partial f(x_0, y_0)}{\partial y} \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 16 \\ -2 \\ -2 \end{bmatrix}$$

. The only way that this can happen is if

$$\frac{\partial f(x_0, y_0)}{\partial x} = 2x_0 + 3y_0 = 8 \quad \text{and} \quad \frac{\partial f(x_0, y_0)}{\partial y} = 3x_0 - 2y_0 = -1.$$

This gives us two equations with two unknowns. In particular, multiplying the first equation by 2 and the second by 3 and adding we have $13x_0 = 13$ so $x_0 = 1$ and then we must have $y_0 = 2$. So the only point where the tangent plane is parallel is at $(x_0, y_0) = (1, 2)$.

9. Consider the function $f(x,y) = 3x^4y - 2x^2y^2 + y^4$.

(a) Write an equation for the contour line (or level curve) for $f(x,y)$ that passes through the point $(1,-1)$.

Contour lines are when $f(x,y) = C$. We need to make sure that we get the contour line that passes through the point $(1,-1)$, and since $f(1,-1) = -4$ then our desired contour line is $f(x,y) = -4$, i.e.,

$$3x^4y - 2x^2y^2 + y^4 = -4.$$

(b) Find two different *unit* vectors $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} c \\ d \end{bmatrix}$ that are perpendicular to the contour line found in (a) at the point $(1,-1)$.

One of the important properties of gradient vectors is that they are *perpendicular* to contour lines. So to find one vector we can find the gradient vector and then normalize it. We have

$$\nabla f(x,y) = \begin{bmatrix} 12x^3y - 4xy^2 \\ 3x^4 - 4x^2y + 4y^3 \end{bmatrix} \quad \text{so} \quad \nabla f(x,y) = \begin{bmatrix} -16 \\ 3 \end{bmatrix}.$$

Since $|\nabla f(x,y)| = \sqrt{(-16)^2 + 3^2} = \sqrt{265}$ then our first unit vector perpendicular to the contour line at $(1,-1)$ is

$$\begin{bmatrix} \dfrac{-16}{\sqrt{265}} \\ \dfrac{3}{\sqrt{265}} \end{bmatrix}.$$

The second unit vector is found by taking the one that goes in the exact opposite direction which can be found by multiplying the vector by -1, i.e.,

$$\begin{bmatrix} \dfrac{16}{\sqrt{265}} \\ \dfrac{-3}{\sqrt{265}} \end{bmatrix}.$$

10. Let $f(x,y) = x^3 - 8xy + 2y^2 - 3x + 4y - 23$.

(a) Find the two critical points for $f(x,y)$.

To find the critical points we solve

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= 3x^2 - 8y - 3 = 0\\ \frac{\partial f(x,y)}{\partial y} &= -8x + 4y + 4 = 0\end{aligned}$$

Taking the second equation and solving for y we have $y = 2x - 1$, which if we substitute into the first equation we have

$$3x^2 - 8(2x-1) - 3 = 3x^2 - 16x + 5 = (3x-1)(x-5) = 0.$$

So that $x = 1/3$ or $x = 5$. Now using $y = 2x - 1$ we get that our two critical points are at

$$\left(\frac{1}{3}, -\frac{1}{3}\right) \quad \text{and} \quad (5,9).$$

(b) Determine if these points are maximums, minimums or neither.

Calculating the second derivatives we have $f_{xx} = 6x$, $f_{yy} = 4$ and $f_{xy} = -8$, so that

$$D(x,y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 24x - 64.$$

Plugging in we have

$$D\left(\frac{1}{3}, -\frac{1}{3}\right) = -56 < 0 \quad \text{so} \quad \left(\frac{1}{3}, -\frac{1}{3}\right) \quad \text{is a saddle, and}$$

$$D(5,9) = 56 > 0 \quad \text{and} \quad f_{xx}(5,9) = 30 > 0 \quad \text{so} \quad \left(5,9\right) \quad \text{is a minimum.}$$

Equation sheet – the following may or may not be helpful.

$$\sin^2 x+\cos^2 x = 1, \qquad \tan^2 x+1 = \sec^2 x, \qquad \ln(ab) = \ln a+\ln b, \qquad \ln a^b = b\ln a$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx+\int_c^b f(x)\,dx, \qquad \frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f\big(h(x)\big)h'(x)-f\big(g(x)\big)g'(x)$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \int x^k\,dx = \frac{1}{k+1}x^{k+1}+C \ \ (k\neq 1), \qquad \int \frac{dx}{x} = \ln x+C$$

$$\int e^x\,dx = e^x + C, \qquad \int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\tan x\,dx = \sec x+C, \qquad \int \sec^2 x\,dx = \tan x+C, \qquad \int \frac{1}{1+x^2}\,dx = \arctan x+C$$

$$\begin{matrix}\text{average}\\ \text{value}\end{matrix} = \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad \begin{matrix}\text{area when}\\ f(x)\geq g(x)\end{matrix} = \int_a^b \big(f(x)-g(x)\big)\,dx$$

$$\begin{matrix}\text{volume of}\\ \text{revolution}\end{matrix} = \pi\int_a^b \big((f(x))^2-(g(x))^2\big)\,dx, \qquad \int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du \ \ (\text{where } u = g(x))$$

$$\int u\,dv = uv - \int v\,du, \qquad \text{Partial fractions}\begin{cases} \text{* check to see if division needed}\\ \text{* factor denominator}\\ \text{* decompose into small pieces}\\ \text{* integrate each piece}\end{cases}$$

$$\begin{matrix}\text{Taylor polynomial for}\\ f(x) \text{ around } x=a\end{matrix} = P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\begin{matrix}\text{solving}\\ \text{differential}\\ \text{equations}\end{matrix}\begin{cases}\text{* separate}\\ \text{* integrate}\\ \text{* simplify}\end{cases} \qquad \frac{dy}{dx} = g(y) \text{ and } g(\widehat{y}) = 0 \text{ then } \begin{cases} g'(\widehat{y}) < 0 & \text{stable}\\ g'(\widehat{y}) > 0 & \text{unstable}\end{cases}$$

$$\begin{matrix}\mathbf{x} = [x_1, x_2, \ldots, x_n]'\\ \mathbf{y} = [y_1, y_2, \ldots, y_n]'\end{matrix} \qquad |\mathbf{x}| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}, \qquad \mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i = |\mathbf{x}||\mathbf{y}|\cos(\theta)$$

$$\begin{matrix}\text{Plane with point } (x_0, y_0, z_0)\\ \text{and normal vector } \mathbf{n} = [a, b, c]'\end{matrix} : \quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}, \quad \begin{matrix}\text{Tangent plane}\\ \text{(linear approximation)}\end{matrix} : \ z = f(x_0,y_0)+\frac{\partial f(x_0,y_0)}{\partial x}(x-x_0)+\frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)$$

$$\nabla f(x,y) = \Big[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\Big]', \qquad D_{\mathbf{u}}(x_0,y_0) = \nabla f(x_0,y_0)\cdot\mathbf{u} \ \ (\mathbf{u} \text{ a unit vector})$$

$$\begin{matrix}\text{critical point}\\ \Leftrightarrow \nabla f = \mathbf{0}\end{matrix}, \qquad D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\big(f_{xy}(x_0,y_0)\big)^2\begin{cases} <0 \text{ then saddle}\\ >0 \text{ and } f_{xx}>0 \text{ then min}\\ >0 \text{ and } f_{xx}<0 \text{ then max}\end{cases}$$

Find min/max of $f(x,y)$ given constraint $g(x,y) = c$

$\Leftrightarrow$ find critical points of $F(x,y,\lambda) = f(x,y) - \lambda\big(g(x,y)-c\big)$