Final Review

Previous material

The final is comprehensive and will cover the whole course. The new material consists of the multivariable portion, and that will be reviewed here. The older material can be found in the other reviews. (Note: Riemann sums will again **not** be on the test.)

Vectors and lines and planes (oh my!)

A point in n-dimensional space is an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$. In n-dimensional space to represent direction (for example, direction between points) we use vectors which again are n-tuples of numbers (we use "$(\cdots)$" for points and "$[\cdots]$" for vectors to distinguish between the two), i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]'$$

For $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ the vector starting at A and going to B is given by $\vec{AB} = [b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n]$.

The length (or magnitude) of a vector $\mathbf{x}$ is

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

If $|\mathbf{x}| = 1$ then $\mathbf{x}$ is called a *unit vector*. Any vector $\mathbf{y} \neq 0$ can be made a unit vector by scaling it to the right length, i.e., $\mathbf{y}/|\mathbf{y}|$.

If $\mathbf{x} = [x_1, x_2, \ldots, x_n]'$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]'$ then the dot product of $\mathbf{x}$ and $\mathbf{y}$, denoted $\mathbf{x} \cdot \mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

If θ is the angle between two vectors (i.e., when putting the "tail" of the vectors together) then

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta \quad \text{so} \quad \theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}\right).$$

In particular we see that we can use dot products to find the angles between vectors. By the above relationship we see that if the angle between the two vectors is a right angle (i.e., $90°$) then the dot product is 0, so we say two vectors are perpendicular if their dot product is 0.

To find a line we need a point $(\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_n})$ and a vector $[a_1, a_2, \ldots, a_n]'$ (the vector giving us the direction. In vector form the line then is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \widehat{x_1} \\ \widehat{x_2} \\ \vdots \\ \widehat{x_n} \end{bmatrix} + t \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

If we break this into the individual entries we get the parametric form of the line

$$\begin{aligned} x_1 &= \widehat{x_1} + t a_1 \\ x_2 &= \widehat{x_2} + t a_2 \\ &\vdots \\ x_n &= \widehat{x_n} + t a_n \end{aligned}$$

A plane in 3-dimensional space is a flat surface that extends out infinitely in all directions. To describe a plane we use a point and a *normal vector* (a vector which is perpendicular to all vectors in the plane). If $\mathbf{n} = [a, b, c]'$ is a normal vector and $r_0 = (x_0, y_0, z_0)$ is a point in the plane then all points $r = (x, y, z)$ in the plane must satisfy

$$\mathbf{n} \cdot (r - r_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Note that given a plane by reading off the coefficients (if they are all on the same side of the equals sign) we can easily find the normal vector.

Functions of several variables

A function is a rule which assigns to each input (in our case a point in n-dimensional space) a single output (in our case a number). The *domain* of the function is the set of all possible points that we can put in the function and get an output. When looking for the domain look for division by 0 problems, square roots of negatives and logs of non-positive numbers. The *range* of the function is the set of all possible outputs.

Given a function we can associate it with a graph. For instance the function $z = f(x, y)$ can be associated with a surface by plotting points $(x, y, f(x, y))$ for all (x, y) in the domain. The graphical representation of a function can give us useful information but is generally hard to find (on the test you will **not** be asked to draw any pictures).

A lot of useful information about a function can be found by taking slices, i.e., we can hold one of the variables constant. We can also take slices in the z-direction and produce *level curves* or *contour lines* (these are similar to the lines which appear on topographical maps). These are found by looking at $f(x, y) = c$ where c is a constant. As c varies this will correspond to different curves in the plane.

Limits and continuity

Limits are useful when dealing with expressions of the form 0/0 (which is undefined), the basic idea is to look at what is happening "nearby" and use that to tell us what "should" happen. In most cases for our class we can plug in the limiting point and if we don't get 0/0 then we're done. In particular, if $f(x, y)$ is a polynomial in x and y then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Suppose that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

then the following hold:

- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(a\,f(x,y)\big) = a\,L$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x,y) + g(x,y)\big) = L + M$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x,y)g(x,y)\big) = LM$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f(x,y)}{g(x,y)}\right) = \frac{L}{M} \qquad (\text{if } M \neq 0)$

A limit does not exist if when we approach the limiting point along different curves we get *different* answers. Typical ways to approach the limiting point are along the x direction, along the y direction, along a straight line, or some other "nice" curve.

A function is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

All polynomials are continuous. If $g(z)$ is continuous and $h(x,y)$ is continuous then so is $g\big(h(x,y)\big)$.

Partial derivatives

To take the derivative of the function with respect to more than one variable we use *partial derivatives* (when we have a function of several variables we use the symbol "∂" instead of "d"). The basic idea is to fix all but one of the variables and then take derivatives as we have before.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Geometrically if we want to find $\frac{\partial f}{\partial x}$ we are finding the tangent line when we look at a slice corresponding to y.

We can take higher order derivatives, i.e.,

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$f_{yx}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

And so forth and so on. Under some simple continuity assumptions (which will always hold for functions we consider) $f_{xy} = f_{yx}$.

Tangent planes and linearization

A tangent plane in multi-variable functions is analogous to tangent lines for single variable functions. For a function $z = f(x,y)$ the tangent plane at (x_0, y_0) is given by

$$z = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0).$$

The tangent plane has the point $(x_0, y_0, f(x_0, y_0))$ (i.e., touches the function at the right point) and locally gives the best linear approximation (i.e., "linear approximation" = "tangent plane"),

$$L(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0).$$

We can use the linear approximation to approximate the function *near* points for which we have information.

Gradients and directional derivatives

The *gradient vector* of the function $f(x,y)$ is a vector where the entries are the partial derivatives. We denote this vector using the symbol "∇", so for example

$$\nabla f(x,y) = \begin{bmatrix} \dfrac{\partial f(x,y)}{\partial x} \\ \dfrac{\partial f(x,y)}{\partial y} \end{bmatrix}.$$

The directional derivative of $f(x,y)$ in the direction $\mathbf{u}$ (where $\mathbf{u}$ is a unit vector) is

$$D_{u} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

This corresponds to the rate of change in the function f if we start at (x,y), turn to the direction given by u and moved forward. Using properties of dot products we have

$$-|\nabla f(x,y)| \le D_{u} f(x,y) \le |\nabla f(x,y)|$$

and in particular the gradient vector points in the direction of greatest increase, the negative of the gradient vector points in the direction of greatest decrease, and the level curve that passes through the point (x,y) is *perpendicular* to the gradient vector.

Optimization

For the function $f(x,y)$, a point (x_0, y_0) is a *local maximum* if $f(x_0,y_0) \ge f(x,y)$ for (x,y) "near" (x_0,y_0). Similarly, a point (x_0,y_0) is a *local minimum* if $f(x_0,y_0) \le f(x,y)$ for (x,y) "near" (x_0,y_0).

These extreme values occur at critical points, which correspond to when $\nabla f(x,y) = \mathbf{0}$ (the vector of all 0's), or put another way when all of the partial derivatives of the function are 0. So to *find* the critical points we set the partial derivatives to 0 and solve (this can be messy!).

To determine what type of extremum this is (i.e., maximum, minimum or saddle (something which occurs when along one slice it looks like a maximum but along a different slice it looks like a minimum)), we use a second derivative test. Let

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - \big(f_{xy}(x, y)\big)^2.$$

We then evaluate D at the critical point and can use that to determine the type of extremum, i.e.,

$$D(x,y) \begin{cases} > 0 \text{ and } f_{xx} > 0 \text{ then a minimum,} \\ > 0 \text{ and } f_{xx} < 0 \text{ then a maximum,} \\ < 0 \text{ then a saddle,} \\ = 0 \text{ inconclusive.} \end{cases}$$

To find extreme values of $f(x, y)$ given that $g(x, y) = c$ we use the method of Lagrange multipliers. The basic idea is that where we achieve an extreme value the level curves of f must be tangent to g, which is equivalent to saying that $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel (or scalar multiples of one another). So to find *where* the extreme values occur we need to have

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = c.$$

We solve this set of equations for points (x, y), finally by plugging them back into the function (and appropriately interpreting them) we get the desired extreme values. Note, that solving these equations is equivalent to finding the critical values of the auxiliary function

$$F(x, y, \lambda) = f(x, y) - \lambda\big(g(x, y) - c\big).$$

We might be asked to find the maximum and minimum values of a function over some region. If the region is closed and bounded and the function is continuous then there *must* be an absolute minimum and an absolute maximum over that region. To locate it we look for maximums and minimums by first finding the critical points in the interior (i.e., $\nabla f = \mathbf{0}$) and then we find extreme values on the boundary (either by parameterization or using Lagrange multipliers). Once we have found our candidate points we plug all of them in, the largest value is the max, the smallest is the min.