More practice problems from 3B

15 variations on a theme

> Integration can find area, and conversely by finding area we can do some integrals.

Example: Using area, find the exact value of

$$\int_0^1 e^{t^2}\,dt + \int_1^e \sqrt{\ln t}\,dt.$$

Example: Find the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(See also M1.2, M1.6)

> We can rewrite integrals by combining them and/or changing the order of integration. Similarly, we can break integrals into pieces and work on each piece separately.

Example: Given that

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_2^1 f(x)\,dx + \int_2^3 f(x)\,dx.$$

Find $\displaystyle\int_1^2 f(x)\,dx$.

Example: Find $\displaystyle\int_{-\pi/2}^{\pi} \sin|x|\,dx$.

(See also M1.2, M1.3, PF1.2, PF1.3, PF2.3)

> The fundamental theorem of calculus tells us that integration and differentiation are two sides of the same coin.

Example: Find $\displaystyle\lim_{h\to 0} \frac{\int_0^h \arctan t\,dt}{h^2}$. (Hint: L'Hospital.)

Example: Given that $\displaystyle\int_0^{2x} f(t)\,dt = e^{x^2} - Ce^x$, find $f(t)$ and C.

(See also M1.4, PF2.1)

> Rewriting integrals turns hard problems into easier problems. (Particularly true when dealing with polynomials and/or trigonometric functions.)

Example: Find

$$\int_0^{\pi} (x\sin x + \cos x)(x\cos x + \sin x)\,dx.$$

(Hint: $\sin x\cos x = \frac{1}{2}\sin 2x$.)

Example: Find $\displaystyle\int \frac{(x+1)(x-1)}{\sqrt{x}}\,dx$.

(See also M1.1, PF1.7)

> Substitution is one of the best tools for simplifying. Look for functions inside of functions.

Example: Find $\displaystyle\int 2te^{\sin(t^2)}\cos(t^2)\,dt$.

Example: Find $\displaystyle\int \sec^2\sqrt{x}\,dx$.

(See also PM2.2, PM2.5, M2.4, M2.5, PF2.4)

> If substitution fails and we have two functions multiplying together, or one function which has a "nice" derivative, we can try integration by parts.

Example: Find $\displaystyle\int_0^1 t^2\ln(t^2+1)\,dt$

Example: Given $\lambda > 0$, show that $\displaystyle\int_0^{\pi} t\sin(\lambda t)\,dt = 0$ only when $\lambda\pi = \tan\lambda\pi$.

(See also PM2.5, M2.4, PF1.4, PF1.5)

> If an integral involves ∞ it is improper. To figure out what is going on approximate it with an integral not involving ∞ and take a limit.

Example: Find $\displaystyle\int_1^{\infty} \frac{\arctan y}{y^2}\,dy$.

Example: Find $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}}\,dx$.

(See also PM2.1, M2.2, PF1.1)

> If we have polynomial/polynomial then (1) do long division; (2) break apart; (3) integrate each part.

Example: Find $\displaystyle\int \frac{1}{x^3+1}\,dx$.

(Hint: $x^3 + 1 = (x+1)(x^2 - x + 1)$.)

Example: Find $\displaystyle\int \frac{\tan x}{\sec^2 x - 4}\,dx$.

(Hint: multiply top and bottom by $\sec x$.)

(See also PM2.2, M2.5)

> Tangent lines give good approximations for the function. Taylor series generalizes this to higher degree polynomials, the key is to remember that coefficients come from derivatives.

Example: Find the degree 2 Taylor series for $f(x) = e^{\sqrt{x}}$ about $x = 1$. Use this to find an approximation for $e^{\sqrt{2}}$.

Example: Given that $P_3(x) = 2 + 5x - 3x^2 + x^3$ is the degree 3 Taylor polynomial for the function $g(x)$ around $x = 2$, find $g'(2)$. (Be careful!)

(See also PM2.3, M2.1, PF1.8, PF2.1)

If we know how the function is changing then (sometimes) we can figure out what the function is. This involves solving differential equations which are done by separating, integrating and simplifying.

Example: Solve the following differential equation

$$\frac{dY}{dt} = 2t(Y^2+1) \quad \text{with} \quad Y(0)=1.$$

Example: On his 30th birthday Mr. Sneebly opened an IRA account which has a return of 5% and started depositing 5000 dollars each year. If M is the number of dollars in his account at time t in years since turning 30 then we have that

$$\frac{dM}{dt} = 5000 + \frac{1}{20}M.$$

If Mr. Sneebly retires at age 65, how much money will he have in his account?

(See also PM2.4, M2.3, PF1.6, PF2.7)

Vectors are used to indicate direction. The most important vector for a plane is the normal vector.

Example: Find a plane that goes through the point $(3,-1,2)$ and is parallel to the tangent plane of $z = x^2 - y^2$ at the point $(2,1,3)$.

Example: When two planes intersect, the angle of their intersection is given by the angle between their normal vectors. Find the angle of intersection between the planes $z = 3x+2y+5$ and $2x+y+z=7$.

(See also PF2.8)

Limits tell us what should happen based on what is happening nearby. If we get two different answers about what should happen then the limit does not exist.

Example: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$. (Hint: $x=y^2$.)

Example: Find $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy^2}{x^2+y^4}$.

(See also PF2.9)

Ideas for one-variable calculus easily translate into multi-variable calculus. Derivatives become partial derivatives. Tangent lines become tangent planes. Critical points (where derivative is 0) becomes critical points (where partial derivatives are 0).

Example: For which point (x,y) is the tangent plane to $f(x,y) = 3x^2 - y^2 + 2xy + 7x$ parallel to the tangent plane to $g(x,y) = x^2 + 2y^2 - 6y + 17$?

Example: Find all of the critical points for

$$h(x,y) = \frac{1}{2}y^2 + y + \frac{1}{2}x^2 + x - xy + \sin x.$$

(See also PF1.8, PF2.10)

The gradient vector is awesome! It can be used to find directional derivatives, indicate direction of fastest increase and decrease, and is perpendicular to level curves.

Example: Find the derivative of the function $f(x,y) = x^3 - 3x^2y + xy$ at the point $(1,2)$ in the direction of the point $(4,-2)$.

Example: Find a vector perpendicular to the curve in the plane $x^3 - 2x + y^2 = 0$ at the point $(1,-1)$. (Hint: make the curve correspond to a level curve of $z = f(x,y)$ for some appropriate $f(x,y)$.)

(See also PF1.9, PF1.10, PF2.11)

To find local max/min find critical points and use second derivative test. If we are finding max/min given constraints use Lagrange multipliers.

Example: Find and classify the critical points for

$$\kappa(x,y) = 2x^3 - 2x^2y + 6xy + y^2 - x^2.$$

Example: You have recently been hired to paint three large non-overlapping dots (a red dot, a blue dot and a green dot) on the side of a building. They have left the design of the dots to you, the only instruction they have is that the total area must be $200\pi\, m^2$. Your contract allows you to charge $3\pi r$ thousand dollars for painting a red circle of radius r, while for a blue circle you charge $4\pi r$ thousand and for a green circle you charge $5\pi r$ thousand. What radii should you choose for the circles to maximize how much you charge?

(See also PF1.10, PF1.11, PF2.12)