Important things I learned in 3A

(but may have forgotten)

The most important thing from the first quarter of calculus is the idea of *limits*. These are used to help us understand what *should* happen for a particular expression at a particular point based on what is happening nearby. We say that $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$, i.e., if we want to be close to L we just need to be close enough to a.

The major application of limits is for finding the slope of tangent lines to curves (and all that entails). To find slopes we usually require two points (x_1, y_1) and (x_2, y_2) and use the slope equation

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

However for tangent lines we only know one point (the point of tangency). To get around this we approximate by secant lines by picking a second nearby point and then letting that point get closer and closer and closer, and looking at the limit (see above) of that process. In particular the slope of the tangent line of $y = f(x)$ can be denoted $f'(x)$ or $\frac{d}{dx}\big(f(x)\big)$ (also known as the derivative) and is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \qquad (\dagger)$$

There are many important rules for derivatives, all of them based on (†).

- **Sum rule:**

$$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$$

- **Scale rule:**

$$\frac{d}{dx}\big(kf(x)\big) = kf'(x)$$

- **Product rule:**

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

- **Quotient rule:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

- **Chain rule:**

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))g'(x)$$

Beyond the basic rules we can also apply (†) to various functions.

$$\begin{aligned}
\frac{d}{dx}\big(c\big) &= 0\\
\frac{d}{dx}\big(x\big) &= 1\\
\frac{d}{dx}\big(x^k\big) &= kx^{k-1}\\
\frac{d}{dx}\big(e^x\big) &= e^x\\
\frac{d}{dx}\big(\ln x\big) &= \frac{1}{x}\\
\frac{d}{dx}\big(\sin x\big) &= \cos x\\
\frac{d}{dx}\big(\cos x\big) &= \sin x\\
\frac{d}{dx}\big(\tan x\big) &= \sec^2 x\\
\frac{d}{dx}\big(\sec x\big) &= \sec x \tan x\\
\frac{d}{dx}\big(\arctan x\big) &= \frac{1}{1+x^2}
\end{aligned}$$

By combining the rules for derivatives along with the above functions we can take the derivative of almost anything we can encounter. (Of course, learning how to apply all these rules took a lot of practice!)

But why did we learn all of these various rules and do so much practice in learning how to use them? Because the derivative has some wonderful applications. Since the derivative gives us the slope of the tangent line it tells us (locally) about the behavior of the function. So for example if $f'(a) > 0$ then the function is *increasing* around a while if $f'(a) < 0$ then the function is *decreasing* around a. So in particular if we wanted to find a highest point or a lowest point on the curve we would look for a place where $f'(a) = 0$ or $f'(a)$ is not defined (these are also called critical points). So using derivatives we can find maximum and minimum values on the curve and where they occur. This has *many* applications in business and science and is well worth the pain of having to have learned all of the rules.

The second derivative $f''(x)$ tells us about the concavity of the curve and can be useful for sketching the curve.

Another nice rule is L'Hospital's rule:

$$\text{If } \lim_{x\to a} \frac{f(x)}{g(x)} \to \frac{0}{0} \text{ then } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Besides derivatives we can also look at anti-derivatives. So for instance if $F'(x) = \cos x$ then an anti-derivative is $F(x) = \sin x$, i.e., a function whose derivative gives the function we started with. These also have applications, but that will have to wait for this quarter.