Math 3B Review

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About the final

- Friday March 20, 3pm-6pm, Lakretz 110
- No notes, no book, no calculator
- Ten questions
  - Five review questions (Chapters 6,7,8)
  - Five new questions (Chapters 9,10)
- No Riemann sums

Integration

The idea behind integration is to slice what we are interested in finding (i.e., area) into little pieces that we can manage. We then add up all of the little pieces and get the result. If we do this for the function $f(x)$ where we slice between $x = a$ and $x = b$ we get the following interpretation.

$$\int_a^b f(x)\,dx = \begin{bmatrix}\text{area above}\\ x\text{-axis}\end{bmatrix} - \begin{bmatrix}\text{area below}\\ x\text{-axis}\end{bmatrix}.$$

Things to watch out for

In cases where our area is made of triangles, squares and/or circles it is usually better to find the integral by finding the appropriate area.

Properties of integrals

- $\int_a^a f(x)\,dx = 0.$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Differentiation and integration are closely connected.

Fundamental theorem of calculus I

If f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(u)\,du \quad \text{then} \quad F'(x) = f(x).$$

Fundamental theorem of calculus II

If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is *any* anti-derivative of $f(x)$.

Application of FTC I: Leibniz's rule

Combining the fundamental theorem of calculus with the chain rule we have

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(u)\,du\right) = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x).$$

Example

Find $\lim_{x\to 0} \dfrac{\int_x^{\sin x} e^u\,du}{x^4}$.

Application of FTC II: Easier integrals

If we know an anti-derivative of $f(x)$ we can now find integrals. Functions that we know antiderivatives for include:

$$x^k, \quad e^x, \quad \sin x, \quad \cos x, \quad \sec^2 x, \quad \sec x \tan x, \quad \frac{1}{1+x^2}$$

Things to try when looking for antiderivative

- Expanding polynomials.
- Using trigonometric identities.
- Breaking into a sum of simpler pieces.

Applications of integration: Area

When finding the area between curves $y = f(x)$ and $y = g(x)$, if $f(x) \geq g(x)$ between $x = a$ and $x = b$ then the area is given by

$$\text{Area} = \int_a^b (f(x) - g(x))\, dx.$$

Things to watch out for

- If no bounds are given then we need to solve for intersection, i.e., $f(x) = g(x)$.
- If the curves intersect or several curves bound the region then break the problem into smaller pieces.
- Sometimes it is easier to do the integral with respect to y.

Applications of integration: Volume

We find volume by adding up the area of cross sections. The only cross sections that we looked at (i.e., could be tested on) are circles, formed by solids of revolution. So if $f(x) \geq g(x) \geq 0$ and we revolve the region between these curves and $x = a$ and $x = b$ around the x-axis, the resulting volume is

$$\text{Volume} = \pi \int_a^b \left((f(x))^2 - (g(x))^2\right) dx.$$

Things to watch out for

- Don't forget to square the functions and be careful when simplifying.
- Volume (as with area) should never be negative.

Applications of integration: Average value

The average value, f_{avg} of the function on the interval $a \leq x \leq b$ is the value so that the rectangle with width $(b - a)$ and height f_{avg} has the same "area" as under the curve, i.e.,

$$\text{Average value} = f_{avg} = \frac{\int_a^b f(x)\,dx}{b - a}$$

Things to watch out for

- Average value can be negative (it can even be 0).
- Mean value theorem tells us that for a continuous function on $[a, b]$ there is some c between a and b so that $f(c) = f_{avg}$.

Substitution rule

Rules for integration come from rules for differentiation. The most important rule for integration is the substitution rule which comes from the chain rule.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{where} \quad u = g(x).$$

For definite integrals we have

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Things to watch out for

- Look for functions inside of functions.
- If you don't see a substitution to make, try rewriting the integral in a different form.

Integration by parts

The product rule for integration becomes integration by parts.

$$\int u\,dv = uv - \int v\,du \quad \text{or} \quad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

This technique is often used when there is a product of two functions; also when there is some function which is hard to integrate but easy to differentiate (i.e., $\ln x$ or $\arctan x$).

Things to watch out for

- Always try substitution before integration by parts.
- Sometimes must be done several times.
- When doing a definite integral sometimes easier to do the indefinite integral to find anti-derivative and then at the end use FTC to evaluate.

Partial fractions

A rational function is a polynomial over a polynomial if we have a rational function then we can use the method of partial fractions.

- First check degrees. If the degree on top is greater than or equal to that on the bottom do long division.
- Next factor the denominator and use partial fraction techniques to break it into small chunks.
- Integrate each chunk.

Things to watch out for

- If it is not a polynomial over a polynomial don't use partial fractions.
- The denominator will always factor as linear and irreducible quadratics.

Partial fractions – the tricky step

$$\frac{(\cdots)}{(ax+b)^k(\cdots)} = \frac{C_1}{ax+b} + \frac{C_2}{(ax+b)^2} + \cdots + \frac{C_k}{(ax+b)^k} + \cdots$$

$$\frac{(\cdots)}{(ax^2+bx+c)^\ell(\cdots)} = \frac{D_1x+E_1}{ax^2+bx+c} + \cdots + \frac{D_\ell x+E_\ell}{(ax^2+bx+c)^\ell} + \cdots$$

To solve for coefficients we first clear the denominators then

- choose “nice” values for x; or
- group coefficients and set them equal.

(Note: the number of coefficients that need to be solved for is the same as the degree of the denominator.)

Improper integrals

An improper integral is one that involves infinity in some way; either bounds involving $\infty / -\infty$ or vertical asymptotes (such as division by 0 or $\ln 0$).

The way to deal with improper integral is to approximate it by an integral which is not improper and then take a limit, i.e.,

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

Things to watch out for

- Watch out for vertical asymptotes (a sneaky way to hide an improper integral).
- It might need to be broken into several parts.
- Sometimes better to first do integral as an indefinite integral
- L'Hospital's rule is useful for evaluating limits.

Taylor polynomials

We can approximate functions by polynomials. To find the "best" polynomials around $x = a$ we use information about derivatives, we call these Taylor polynomials.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We can use these polynomials to approximate the function for points near $x = a$.

Things to watch out for

- Make sure that the coefficients are numbers, i.e., don't forget to evaluate.
- Given a function we can find the Taylor series, but we also can take a Taylor series and say something about the function.

Differential equations

Differential equation relates how a dependent variable is changing (i.e., the derivative) in terms of the dependent variable and the independent variable. These are very important in physics, biology, chemistry, economics, and getting a good grade in Math 3B.

We restrict ourselves to solving separable differential equations,

$$\frac{dy}{dx} = f(x)g(y) \quad \text{with} \quad y(x_0) = y_0.$$

Method to solve:

1. Separate, put all y terms on one side, x on the other.
2. Integrate, integrate each side.
3. Simplify, solve for the constant and (perhaps) solve for dependent variable.

(more) Differential equations

Things to watch out for

- We can solve for the constant and then simplify for the dependent variable or vice-versa
- We can use the differential equation to solve for a x that y will hit a certain value; or we can see what the value of y will be at time x; and so on.

If $dy/dx = g(y)$ then an equilibrium solution is a solution to $g(y) = 0$. A stable equilibrium is one that attracts points nearby while an unstable equilibrium is one that repels points nearby. There are two methods to test stability of equilibrium solution $\hat{y}$:

- $g'(\hat{y}) < 0$ is stable; $g'(\hat{y}) > 0$ is unstable
- Draw a picture.

Vectors

A vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]'$ can be used to indicate a direction in space. Given points $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ the vector that starts at A and points to B is found by

$$\vec{AB} = [b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n]'.$$

The length (or magnitude) of a vector is found by

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

A vector which has length 1 is called a unit vector (these are often used when dealing with direction).

Dot product

Given vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]'$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]'$ the dot product is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = |\mathbf{x}||\mathbf{y}| \cos \theta,$$

where θ is the angle between the two vectors (if they emanated from the same point). We can use this to solve for the angle between vectors.

Two vectors are perpendicular if the angle they form a right angle. Equivalently two vectors are perpendicular if their dot product is 0.

Lines

To find a line we need a point $(\hat{x}_1, \hat{x}_2, \ldots \hat{x}_n)$ and a direction which will be provided for us by a vector $[a_1, a_2, \ldots, a_n]'$. Then points on the line satisfy

$$\text{Vector form:} \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{bmatrix} + t \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

or

$$\text{Parametric form:} \quad \begin{aligned} x_1 &= \hat{x}_1 + ta_1 \\ &\vdots \\ x_n &= \hat{x}_n + ta_n \end{aligned}$$

Planes

A plane can be thought of as a flat 2-dimensional surface embedded in three dimensional space. The most important part about the plane is a normal vector, or a vector which is perpendicular to all points in the plane. If a plane has the point (x_0, y_0, z_0) and normal vector $\mathbf{n} = [a, b, c]'$ then the set of points (x, y, z) which are in the plane are those which satisfy

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Things to watch out for

- Normal vectors are not unique, they can change by scaling by a constant.
- Two planes are parallel if their normal vectors are parallel (i.e., scalar multiples of each other).
- When reading coefficients to get the normal vector make sure all the coefficients are on the same side.

Functions of several variables

The domain of the function is the set of all possible inputs. Generally when looking for domain we avoid problems, the type of problems that we might have include

$$\frac{\cdots}{0}, \quad \sqrt{<0}, \quad \ln(\leq 0).$$

The range of the function is the set of all possible outputs. Generally speaking this can be very hard to find (one of the reasons Calculus was invented was to help answer such questions).

To understand a function we often look at slices in either x, y or z direction. In the z direction this means $f(x, y) = C$ which are curves in the plane called level curves or contour lines.

Limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Limits don't tell us what does happen, rather they tell us what should happen based on what is happening nearby.

For most functions we can plug in the limit point and evaluate; if we don't get $0/0$ then we're done. If we do get $0/0$ then more work is ahead.

If a limit exists then we should always get the same answer regardless of which path we take to get to the limit point. So if along two different curves we get two different values then the limit does not exist. (Best curves are usually straight lines or curves of the form $y = x^k$, $x = y^\ell$.)

Continuity

A function is continuous at a point if what should happen is what does happen, i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

All polynomials are continuous, and composition of continuous functions are continuous.

Partial derivatives

For multivariable functions we often use partial derivatives (denoted "∂" instead of "d"). Idea is to treat all but one variable as a constant and then take derivatives.

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

We can also take higher order derivatives, i.e., f_{xx}, f_{xy}, f_{yx}, f_{yy} and so on. For "nice" functions (i.e., functions in our class),

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

Tangent planes

A tangent plane is a higher dimensional analog of a tangent line. For a function $f(x, y)$ the tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

Since tangent planes make "good" approximations locally we can use tangent planes to approximate the function.

Things to watch out for

- Make sure to evaluate the partial derivatives.

Gradients

The gradient vector is a vector where the entries are partial derivatives.

$$\nabla f(x,y) = \begin{bmatrix} \dfrac{\partial f(x,y)}{\partial x} \\ \dfrac{\partial f(x,y)}{\partial y} \end{bmatrix}$$

We can use the gradient to find directional derivatives at a point in a direction **u** (a unit vector). Namely,

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

Things to watch out for

- Make sure **u** is a unit vector!

Properties of gradients

At a point (x, y) we have,

$$-|\nabla f(x,y)| \leq D_{\mathbf{u}} f(x,y) \leq |\nabla f(x,y)|.$$

So the gradient bounds the directional derivatives!

The gradient points in the direction of greatest increase.
Opposite the gradient points in the direction of greatest decrease.

The gradient vector $\nabla f(x, y)$ is perpendicular to the level curve that passes through the point (x, y).

Local maximum/minimum

A local maximum (or minimum) is a point which is greater than (less than) all other points nearby. Local maximum/minimum occur at critical points which are where $\nabla f(x, y) = \mathbf{0}$.

To determine if it is a maximum, a minimum or a saddle we use the second derivative test.

$$\text{Let} \quad D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - \left(f_{xy}(x, y)\right)^2$$

$$\text{then} \quad D(x, y) \begin{cases} > 0 & \text{and } f_{xx} < 0 \text{ then a maximum,} \\ > 0 & \text{and } f_{xx} > 0 \text{ then a minimum,} \\ < 0 & \text{then a saddle,} \\ = 0 & \text{inconclusive.} \end{cases}$$

Lagrange multipliers

When maximizing/minimizing $f(x, y)$ given a constraint $g(x, y) = c$ then maximum/minimum occur when

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = c.$$

Equivalently, the critical points of

$$F(x, y, \lambda) = f(x, y) - \lambda\big(g(x, y) - c\big).$$

Find the points, then plug them in to find the extreme values.

Things to watch out for

- The equations are almost always nonlinear (i.e., tricky to solve).
- If nothing else, solve for λ and get relationships among variables.