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MATH 180 (Butler)

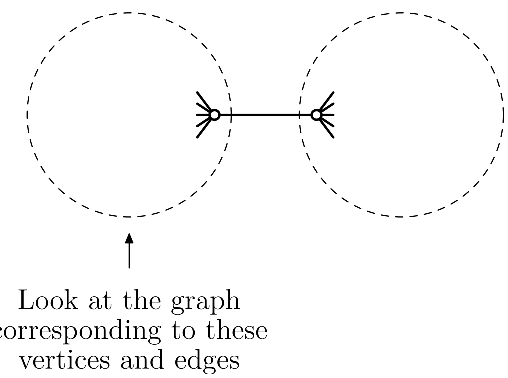
Midterm II, 18 May 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work and (briefly) explain your approach (legibly!). Each problem is worth 10 points. Pick 4 out of the 5 problems to do. (If you do all 5 we will use the best 4 scores.)

1. A *bridge* in a connected graph is an edge whose removal from the graph causes the graph to be disconnected.

(a) Show if a graph is regular of degree k, where k is even then the graph does **not** have a bridge.

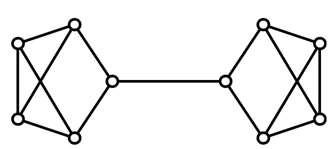
Suppose that we have a bridge, i.e., then our picture is like the following.



Where we have vertices on the left and right and one edge going between the two sets. Examining the graph found by looking at the vertices and edges of the left part we see that we will have one vertex of degree $k-1$ (odd) and all other vertices would have degree k (even). But this is impossible because the number of vertices with odd degree must be even. Therefore our assumption that we have a bridge must be wrong, and so there is no bridge.

(b) Give an example of a graph with 10 vertices which is regular of degree 3 and has a bridge.

Shown below is the only possible graph satisfying the conditions that is also simple.



2. Solve the recurrence for a_n where $a_1 = 2$ and

$$a_n = \frac{2n}{n-1}a_{n-1} + n2^n \quad \text{for } n \geq 2.$$

We start by dividing both sides by n. This gives us

$$\frac{a_n}{n} = 2\frac{a_{n-1}}{n-1} + 2^n.$$

If we now substitute $b_n = a_n/n$ then this translates into the recurrence

$$b_n = 2b_{n-1} + 2^n,$$

with initial condition $b_1 = a_1/a = 2$. The homogeneous portion gives us $r = 2$ so that we have $b_n = A2^n$. For the nonhomogeneous portion, since 2 is a root we will guess $b_n = Cn2^n$, putting this in we have

$$Cn2^n = 2C(n-1)2^{n-1} + 2^n = Cn2^n + (1-C)2^n \quad \text{so} \quad C = 1.$$

Therefore the solution for $b_n = A2^n + n2^n$. Since $b_1 = 2A + 2 = 2$ we can conclude that $A = 0$. So we have

$$a_n = nb_n = n\big(n2^n\big) = n^2 2^n.$$

3. How many ways are there to arrange the numbers 1 to n if you cannot have any occurrence of an i immediately followed by $i+1$ for $i = 1, 2, \dots, n-1$? So for example 13524 is fine but 5$\underline{34}$21 is not.

Let A_i be the set of ways to arrange the numbers so that i is immediately followed by $i+1$. Then the problem asks us to find

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}}\right| = |\mathcal{U}| + \sum_{\substack{I \subseteq [n-1] \\ I \neq \emptyset}} (-1)^{|J|} \left| \bigcap_{j \in I} A_j \right|.$$

The term on the right being the inclusion-exclusion formula.
We first claim

$$\left| \bigcap_{j \in I} A_j \right| = (n - |I|)!.$$

To see this, let us start with the number $1, 2, \dots, n$ all laid down in a row. For every $j \in I$ we need to make sure that j will always be immediately followed by $j+1$ and this can be done by "gluing" j and $j+1$ together and then when we arrange they always stay in a group. We do this for each $j \in I$ and every time we glue two numbers together we are reducing the number of terms that will be rearranged by 1, and so after gluing the elements of I together we have $n - |I|$ terms to rearrange which can be done in $(n - |I|)!$ ways.
Now we use this to rewrite the inclusion-exclusion formula, where instead of summing over I we sum over $k = |I|$, i.e., we sum over the sizes of the subsets and not the subsets themselves. In particular, we have

$$|\mathcal{U}| + \sum_{\substack{I \subseteq [n-1] \\ I \neq \emptyset}} (-1)^{|J|} \left| \bigcap_{j \in I} A_j \right| \;=\; n! + \sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^k (n-k)! \;=\; \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (n-k)!$$

The term $\binom{n-1}{k}$ comes from the fact that there are $\binom{n-1}{k}$ ways to form a subset of size k from $[n-1]$.

4. In this problem we will use an exponential generating function for working with a recurrence. For a sequence $a_0, a_1, a_2, \ldots$ the exponential generating function is $y = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

(a) What is the coefficient of $\frac{x^n}{n!}$ in $y' = \frac{d}{dx}(y)$?

We have

$$y' = \frac{d}{dx}\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right) = \sum_{n\geq 0} a_n n \frac{x^{n-1}}{n!} = \sum_{n\geq 1} a_n \frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 0} a_{n+1} \frac{x^n}{n!}.$$

(We can drop the $n = 0$ term at the third step since it will automatically be 0. The last step comes from shifting the index.) In particular we see that the coefficient of $x^n/n!$ is a_{n+1}.

(b) Let a_n be the zig-zag numbers which count the rearrangements of $1, 2 \ldots, n$ as $b_1 b_2 \cdots b_n$ such that $b_{2k} > b_{2k+1}$ and $b_{2k-1} < b_{2k}$. So $a_0 = 1$ ($\emptyset$), $a_1 = 1$ (1), $a_2 = 1$ (12), $a_3 = 2$ (132, 231) and $a_4 = 5$ (1324, 1423, 2314, 2413, 3412). It is known (you do **not** need to prove this) that for $n \geq 1$

$$2a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k}.$$

Use this recurrence relationship to set up a differential equation for y (be careful with the first term!). (Hint: Recall that $\left(\sum a_n \frac{x^n}{n!}\right)\left(\sum b_n \frac{x^n}{n!}\right) = \sum c_n \frac{x^n}{n!}$ where $c_n = \sum \binom{n}{k} a_k b_{n-k}$.)

First let us figure out what happens for the first term (i.e., $n = 0$). We have $2a_{0+1} = 2a_1 = 2$ and $\sum_{k=0}^{0} \binom{0}{k} a_k a_{n-k} = \binom{0}{0} a_0 a_0 = 1$, in particular they are off by 1. So we have

$$2\sum_{n\geq 0} a_{n+1}\frac{x^n}{n!} = 2a_1 + \sum_{n\geq 1} 2a_{n+1}\frac{x^n}{n!} = 1 + \sum_{k=0}^{0}\binom{0}{k} a_k a_{0-k} + \sum_{n\geq 1}\left(\sum_{k=0}^{n}\binom{n}{k} a_k a_{n-k}\right)\frac{x^n}{n!}$$
$$= 1 + \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} a_k a_{n-k}\right)\frac{x^n}{n!}$$

By part (a) we have that the term on the left is $2y'$ and by the hint we have that the term on the right is $1 + y^2$ and so the desired differential equation is

$$2y' = y^2 + 1.$$

(c) Show that $y = \sec x + \tan x$ satisfies the differential equation found in part (b). Hence the even terms a_{2n} or zig numbers are also called secant numbers, while the odd terms a_{2n+1} or zag numbers are also called tangent numbers. (Hint: $\sec^2 x = \tan^2 x + 1$.)

Using the the derivative of $\sec x$ is $\sec x \tan x$ and the derivative of $\tan x$ is $\sec^2 x$ along with the hint we have

$$\begin{aligned} 2y' &= 2\big(\sec x \tan x + \sec^2 x\big) &&= 2\sec x \tan x + 2\sec^2 x \\ y^2 + 1 &= \big(\sec x + \tan x\big)^2 + 1 &&= \sec^2 x + 2\sec x\tan x + \tan^2 x + 1 = 2\sec x \tan x + 2\sec^2 x. \end{aligned}$$

(To verify that this is the correct solution we should also check the initial conditions. We have that $y(0) = a_0 = 1$ while $\sec 0 + \tan 0 = 1$. So we do have $y = \sec x + \tan x$.)

5. Show that in a connected planar graph there must be a vertex of degree at most five.
(Hint: recall that $|E| \leq 3|V| - 6$.)

Suppose that there is a graph which is planar and with all vertices having degree six or more. Then by the handshake theorem we have

$$2|E| = \sum_{v \in V} d(v) \geq \sum_{v \in V} 6 = 6|V| \quad \text{or} \quad |E| \geq 3|V|.$$

But since it is planar we also have

$$|E| \leq 3|V| - 6.$$

Combining these two we would have to have

$$3|V| \leq |E| \leq 3|V| - 6,$$

which is of course impossible. Therefore if our graph is planar we cannot have all vertices with degree six or more, i.e., there must be a vertex of degree at most five.