Midterm II Material

Mathematical induction is a useful method of proving statements for $n = 1, 2, \ldots$ (though we don't have to start at 1, we can start at any value). In general we need to prove two parts:

1. The statement holds in the first case(s).
2. Assuming the statement holds for the cases $k \le n$, show that the statement also hold for $n+1$.

(This is what is known as *strong induction*. Usually it will suffice for us to only assume the previous case, i.e., assume the case $k = n$ is true and then show $n+1$ also holds (which is known as *weak induction*.)

The *most* important thing about an induction proof (and also for recurrences) is to see how to relate the previous case(s) to the current case. Often once that is understood the rest of the proof is very straightforward.

A recurrence relation (or recursion relation) is a sequence a_n where the nth term a_n is expressed as a function of n and the previous a_k. For example the Fibonacci numbers are based on the recurrence relation $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$ (we need to have terms F_0 and F_1 to "prime the pump" of our recurrence, i.e., just the recurrence relationship is not enough to evaluate we also need some starting values).

We are interested in *solving a recurrence* which is to find an expression for a_n that is independent of the previous terms. In other words we want to find some function $f(n)$ so that $a_n = f(n)$. One method to do this is to look at the first few terms, and find a pattern. If we think we have the pattern then to verify we need to verify the initial conditions and recurrence relations are satisfied (closely analogous to induction).

The key to setting up a recurrence for a given problem is to relate what you are counting for the current case to what you counted in earlier cases. Often the idea is to look for how to add/remove the last object (like we did for derangements) or look for ways to decompose into smaller parts (like we did for Dyck paths). Also once we have a recurrence sometimes we can manipulate it into a more convenient form (like we did for compositions).

Some recurrences can be solved systematically. For instance homogeneous constant coefficient linear recursions of order k, i.e.,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

can be solved by guessing that our solution is of the form $a_n = r^n$, the recurrence then puts a restriction on r, namely,

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

We solve for the roots of this polynomial $r_1, r_2, \ldots, r_k$ to get solutions, and then do a linear combination of these solutions to get our general answer,

$$a_n = D_1 r_1^n + D_2 r_2^n + \cdots + D_k r_k^n.$$

The $D_1, D_2, \ldots, D_k$ are then determined by the initial conditions. Suppose we have a repeated root, ρ of multiplicity ℓ, then we have to modify the solution. Our method to do this is to introduce powers of n (nudging our solutions), so that our solution will look like

$$a_n = D_1 \rho^n + D_2 n \rho^n + D_3 n^2 \rho^n + \cdots + D_\ell \rho^{\ell-1} \rho^n + \cdots$$

where the "$+\cdots$" corresponds to the contribution of any other roots.

If we do not have constant coefficients (or are not linear) we can sometimes modify the recurrence to become linear, i.e., the recurrence $a_n^2 = 4a_{n-1}^2 - 4a_{n-2}^2$ becomes $b_n = 4b_{n-1} - 4b_{n-2}$ after the substitution $b_n = a_n^2$. We then solve as before for b_n and then finally substitute back to get a_n.

If we are dealing with a non-homogeneous recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n).$$

where $f(n) = \sum(\text{polynomial})\cdot\rho^n$ (i.e., $f(n)$ is similar to the form of homogeneous solutions) then we can solve it using the following steps.

1. Solve the homogeneous part (i.e., the recurrence without the $f(n)$).
2. Solve the non-homogeneous part by setting up a solution for a_n with some coefficients to be determined by the recurrence.
3. Combine the above two part to get the general solution. Solve for the constants using the initial conditions.

Note that the order of operations is important. That is, we need to solve for the homogeneous part before we can do the non-homogeneous part and we need to solve both parts before we can use initial conditions.

The reason that we need to solve the homogenous part first is that it can influence how we solve the non-homogeneous part. So now let us look at step 2 a little more closely. So suppose that we have

$$f(n) = (j\text{th degree polynomial in } n)\rho^n.$$

We look at the homogeneous part and see if ρ is a root, i.e., part of the homogeneous solution. If it is not a root then we guess that the non-homogeneous solution will be of the form

$$a_n = (B_j n^j + B_{j-1} n^{j-1} + \cdots + B_0)\rho^n,$$

where $B_j, B_{j-1}, \cdots, B_0$ are constants which will be determined by putting this into the recursion, grouping coefficients and then making sure each coefficient is zero. (See the examples below.)

If ρ is a root then part of the above guess actually is in the homogeneous part and cannot contribute to the non-homogeneous part. In this case we need to gently nudge our solution. To do this, suppose that ρ occurs as a root m times. Then we modify our guess for the non-homogeneous solution so that it is now of the form

$$a_n = (B_j n^{m+j} + B_{j-1} n^{m+j-1} + \cdots + B_0 n^m)\rho^n.$$

That is we multiply by a power of n^m to push the terms outside of the homogeneous solution.

If $f(n)$ has several parts added together we essentially do each part seperately and combine them together.

We can use generating functions to help solve recurrences. The idea is that we are given a recurrence for a_n, and we want to solve for a_n. This is done by letting

$$g(x) = \sum_{n\geq 0} a_n x^n,$$

we then translate the recurrence into a relationship for $g(x)$ which lets us solve for $g(x)$. We finally take the function $g(x)$ and use it to find the coefficient for x^n.

Suppose we have initial conditions $a_0, a_1, \ldots, a_{k-1}$ and a recurrence $a_n = f(n, a_{n-1}, a_{n-2}, \ldots)$ for $n \geq k$.

1. Write $g(x) = \sum_{n\geq 0} a_n x^n$.

2. Break off the initial conditions and use the recurrence to replace the remaining terms, i.e., so we have
$$g(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + \sum_{n\geq k} f(n, a_{n-1}, a_{n-2}, \ldots)x^n.$$

3. (Hard step!) rewrite the right hand side in terms of $g(x)$ and/or other functions.

4. Now solve for $g(x)$, and then expand this into a series to read off the coefficient to get a_n. For example, this can be done using general binomial theorem or partial fractions.

Sometimes we might only be interested in solving for the function $g(x)$.

Some things to watch for are shifting the indices of a sum, moving x's in and out of the sums, multiplying two sums together, breaking up addition, and manipulating some common sums. It is good to be able to manipulate the sum

$$\sum_{k\geq 0} x^k = \frac{1}{1-x}$$

as well as

$$\sum_{k\geq 0} \frac{x^k}{k!} = e^x.$$

We now turn to set theory, our setting is that we have a universe of elements, $\mathcal{U}$ and some collection of subsets. If A is a subset and $\overline{A}$ is the complement of A (everything not in A) and $|A|$ denotes the number of elements in A then

$$|\mathcal{U}| = |A| + |\overline{A}|.$$

For two sets A and B we let $A \cup B$ denote the union of A and B where

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\},$$

and $A \cap B$ denote the intersection of A and B where

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}.$$

de Morgan's laws

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} \\ \overline{A \cup B} &= \overline{A} \cap \overline{B} \end{aligned}$$

If we want to count how many elements are in $|A \cup B|$ then we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The idea being that if we only look at $|A| + |B|$ then we overcount elements that are in both so we need to correct for this. If we are dealing with three sets then the analogous statement is

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

These are simple examples of the more general inclusion-exclusion principle.

Inclusion-Exclusion Principle:
Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| = |\mathcal{U}| + \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|} \left|\bigcap_{j\in I} A_j\right|.$$

Inclusion-Exclusion Principle II:
Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then

$$\left|A_1 \cup A_2 \cup \cdots \cup A_n\right| = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|-1} \left|\bigcap_{j\in I} A_j\right|.$$

These are proved by verifying that each element $x \in \mathcal{U}$ gets counted (in total) the same on each side establishing the equality. These inclusion-exclusion principle applications are mostly used when

$$\left|\bigcap_{j\in I} A_j\right|$$

depends only on $|I|$ and not the actual elements of I. In this case we can rewrite the sum not over all subsets I but over the size of all subsets, i.e., $k = 1, 2, \ldots, n$, and then add an additional factor of $\binom{n}{k}$ (the number of subsets I with k elements).

A *graph* G consists of two sets, a vertex set V and an edge set E, we sometimes will write this as $G = (V, E)$. The vertex set represents our objects and the edge set represents our connections between objects, we will visually denote vertices by "". The edge set can have several different ways of describing how we connect vertices which leads to different types of graphs. We will, for now, restrict our attention to the case when the set E consists of unordered pairs of vertices, i.e., $\{u, v\}$. These graphs are known as *undirected graphs*. Visually we represent the edges as "".

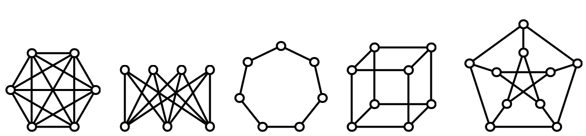
Further we will restrict ourselves to *simple* undirected graphs. Simple graphs refer to graphs without loops or multiple edges. A loop is an edge that connects a vertex to itself, i.e., . A multiple edge is several edges going between the same two vertices, i.e., . In other words we want to eliminate the possibility of having a repeated element in our sets.

A *bipartite* graph is a graph there the vertices can be split into two parts $V = U \cup W$ where U and W are disjoint and such that all the edges in the graph connect a vertex in U to a vertex in W.

A graph is bipartite if and only if every cycle in the graph has even length.

A cycle in the graph is a sequence of vertices $v_1, v_2, \ldots, v_n$ where $v_i \sim v_{i+1}$ (i.e., v_i is adjacent to v_{i+1}) for $i = 1, \ldots, n-1$ and $v_n \sim v_1$. A matching of a graph is a subset of edges so that each vertex is in exactly one edge. (Note that this definition does not require the graph is bipartite.)

Some important graphs are K_n (the complete graph on n vertices), $K_{m,n}$ (the complete bipartite graph with parts of size m and n), C_n (the cycle on n vertices), P_n (the path on n vertices), Q_n (the hypercube on 2^n vertices), and of course the Petersen graph. (Below are K_6, $K_{3,4}$, C_7, Q_3 and the Petersen graph.)



Two vertices u and v are *adjacent* if there is an edge connecting them, this is denoted as $u \sim v$. An edge e and a vertex v are *incident* if the vertex v is contained in the edge e (pictorially the edge connects to the vertex). The *degree* of a vertex is the number of edges incident to the vertex, equivalently the number of vertices adjacent to the vertex (assuming we do not allow loops and multiple edges), we denote the degree of a vertex by d_v or $d(v)$. A graph where all of the vertices have the same degree is known as a *regular* graph. The Petersen graph is a regular graph of degree 3, the hypercube is a regular graph of degree n, the complete graph K_n is a regular graph of degree $n-1$ and the cycle C_n is regular of degree 2.

Handshake Theorem:
For a graph G let $|E|$ denote the number of edges, then

$$\sum_{v\in V} d(v) = 2|E|.$$

In particular this implies that the sum of the degrees must be an even number. So we have the following immediate corollary.

Corollary of the Handshake Theorem:
The number of vertices with odd degree must be even.

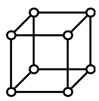
We say two graphs G and H are *isomorphic* if there is a bijective (one-to-one/onto) map $\phi : V(G) \to V(H)$ so that $u \sim v$ in G if and only if $\phi(u) \sim \phi(v)$ in H. Or in other words we can map the vertices of one graph to the other in a way that preserves adjacency.

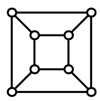
To show that two graphs are isomorphic then it suffices for us to find a way to relabel the vertices of one graph to produce the other graph. To find a rule for relabeling we often try to identify some vertices that must be mapped one to the other (i.e., because of degree considerations) and slowly build up the relationship.

To show that two graphs are not isomorphic we need to find some structure that one graph has that the other does not, showing they cannot be the same graph. For example number of vertices, number of edges, subgraphs, complements, connected, planar, etc.

A graph is *connected* if between any two vertices we can get from one vertex to another vertex by going along a series of edges. That is for two vertices u, v there is a sequence of vertices $v_0, v_1, \ldots, v_k$ so that $u = v_0 \sim v_1$, $v_1 \sim v_2$, …, $v_{k-1} \sim v_k = v$.

A graph is *planar* if it can be drawn in the plane in such a way so that no two edges intersect. Of course every graph can be drawn in the plane, the requirement that edges do not intersect though adds some extra constraint. It is important to note in this definition that a graph is planar if there is *some* way to draw it in the plane without edges intersecting, it does not mean that every way to draw it in the plane avoids edges intersecting. So for example in the we can draw Q_3 as shown below on the left. In this drawing we have edges crossing each other twice so that this is not a way to embed it in the plane without edges crossing, but we can draw it in a way without edges crossing which is shown below on the right.





Not all graphs are planar. For example K_5 and $K_{3,3}$ cannot be drawn in the plane without having edges cross. In some sense these are *the* problem graphs for planar graph.

> *Kuratowski's Theorem:*
> A graph is planar if and only if it does not contain a topological K_5 or $K_{3,3}$.

To say that a graph G contains a topological K_5 or $K_{3,3}$ means that there is a subgraph of G that by contracting edges (i.e., if $u{\sim}v$ is an edge remove the edge and combine u and v into one new vertex) produces K_5 or $K_{3,3}$. So for example the Petersen graph is not planar since we can contract the edges of the outer cycle to the inner star to get a K_5.

If a graph is planar then we can draw it in the plane, this then divides the plane up into a finite number of regions called faces. If the graph were drawn on a piece of paper and we cut along the edges then the resulting different pieces of papers are the faces. For example for the drawing of Q_3 above there are six faces (don't forget to count the outer face!). If we let V denote the set of vertices, E the set of edges and F the set of faces for a drawing of a planar graph then we have the following.

> *Euler's formula:*
> For a connected planar graph
> $$|V| - |E| + |F| = 2.$$

Note in particular that the number of faces does not depend on how we draw the graph in the plane.

Euler's formula is *the* main tool for dealing with planar graphs (so if a problem asks a question about planar graphs a good place to start is with Euler's formula or a variation). For example if a connected planar graph has at least three vertices then *every* face must have at least three edges. We can count the number of edges by adding up how many edges are in each face, since each edge will be in exactly two faces this will count all edges twice, so we have

$$2|E| = \sum_{f\in F} \left| \begin{matrix} \text{edges in} \\ \text{face } f \end{matrix} \right| \geq \sum_{f\in F} 3 = 3|F|,$$

so that $|F| \leq (2/3)|E|$. Substituting this in and simplifying we have

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| \quad \text{or} \quad |E| \leq 3|V| - 6.$$

This must hold for every connected planar graph, and so we can use this to conclude that K_5 cannot be planar since it has $|E| = 10$ and $3|V| - 6 = 9$ so it does not satisfy the relationship. A similar argument shows that $K_{3,3}$ is not planar.

One useful technique for proving statements is to use *proof by contradiction*. The idea is to assume that the statement does not hold and show that this leads to an impossible situation, and so our assumption that the statement is false is wrong so the statement must be true. (Or put more succinctly we are proving something is not not true.)

As an example suppose we want to show that every undirected simple graph with two or more vertices has two vertices of the same degree. We suppose that this is false, so that there is an undirected simple graph with all vertices having distinct degrees. If this graph had n vertices then it must have a vertex of degree 0 and a vertex of degree $n-1$. But this is impossible because the vertex of degree 0 is not connected to anyone while the vertex of degree $n-1$ is connected to everyone, a contradiction. So our assumption that there is a graph with distinct degrees is false, so every graph must have some two vertices with the same degree.

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