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MATH 180 (Butler)

Midterm I, 22 April 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work and (briefly) explain your approach (legibly!). Each problem is worth 10 points.

1. Find the probability that if we distribute eight identical oranges and five distinct apples (i.e., one Gala, one Red Delicious, one McIntosh, one Braeburn and one Fuji) among three people that each person will have at least one orange and at least one apple.

We need to count two things, the number of outcomes with the restriction that there is at least one orange and at least one apple per person, and the number of possible outcomes.

To count the first thing we need to distribute the 8 oranges. To make sure that everyone gets at least one orange we first give one orange to each person. This leaves 5 oranges (balls) to distribute among 3 people (bins) which can be done in $\binom{5+3-1}{3-1} = \binom{7}{2}$ ways. We now distribute the apples. Since each person must have at least one then we can distribute apples in one of the following ways: $(1,2,2)$, $(2,1,2)$, $(2,2,1)$, $(1,1,3)$, $(1,3,1)$ and $(3,1,1)$. In the first three cases this can be done in $\frac{5!}{1!2!2!}$ ways while in the last three cases this can be done in $\frac{5!}{1!1!3!}$ ways. Combining everything together we have that the total number of ways that everyone gets at least one orange and at least one apple is

$$\binom{7}{2}\left(3\cdot\frac{5!}{1!2!2!} + 3\cdot\frac{5!}{1!1!3!}\right).$$

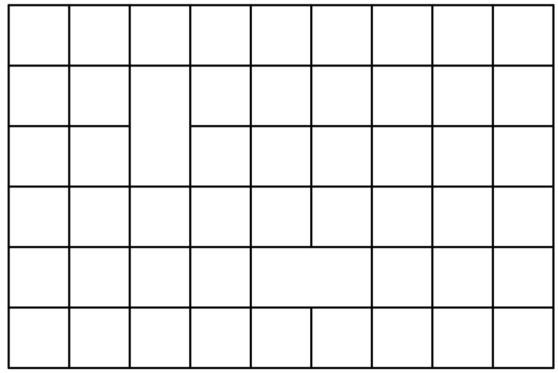
To count the second thing we need to distribute 8 oranges (balls) among 3 people (bins) which can be done in $\binom{8+3-1}{3-1} = \binom{10}{2}$ ways. To distribute the apples, each apple can go to one of three people so that there are 3^5 ways to distribute the apple. Combining everything together we have that the total number of ways to distribute apples and oranges is

$$\binom{10}{2}3^5.$$

Combining the two the probability that each person will have at least one apple and at least one orange is

$$\frac{\binom{7}{2}\left(3\cdot\frac{5!}{1!2!2!} + 3\cdot\frac{5!}{1!1!3!}\right)}{\binom{10}{2}3^5} = \frac{3150}{10935} = \frac{70}{243} = 0.28806584\ldots.$$

2. Consider the following grid.



(a) How many walks are there from the lower left corner to the upper right corner taking only *up* steps and *right* steps.

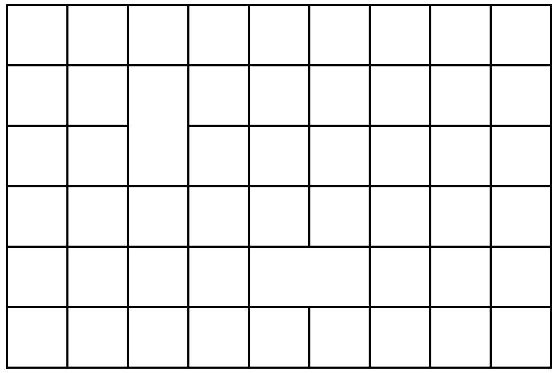
Let us label the bottom left corner as $(0,0)$ and the upper right corner as $(9,6)$. The total number of walks from the bottom left to the upper right is $\binom{9+6}{6} = \binom{15}{6}$ (i.e., we need to choose when to make the 6 up steps among the 15 steps).

We now need to take off the walks that were using the edges that are now missing. If a walk uses the edge $(2,4) \to (3,4)$ then we first need to get to $(2,4)$ (which can be done in $\binom{4+2}{4} = \binom{6}{4}$ ways), then move to $(3,4)$ (which can be done in 1 way), then finally more to $(9,6)$ (which can be done in $\binom{6+2}{2} = \binom{8}{2}$ ways). This makes $\binom{6}{4}\binom{8}{2}$ ways using the edge $(2,4) \to (3,4)$.

Similarly to count the walks using the edge $(5,1) \to (5,2)$ we first need to get to $(5,1)$ (which can be done in $\binom{5+1}{1} = \binom{6}{1}$ ways) then move to $(5,2)$ (which can be done in 1 way), then finally move to $(9,6)$ (which can be done in $\binom{4+4}{4} = \binom{8}{4}$ ways). This makes $\binom{6}{1}\binom{8}{4}$ ways using the edge $(5,1) \to (5,2)$.

Combining the number of walks is

$$\binom{15}{6} - \binom{6}{4}\binom{8}{2} - \binom{6}{1}\binom{8}{4} = 4165.$$



(b) How many walks are there from the upper left corner to the lower right corner taking only *down* steps and *right* steps.

We use the same technique as in the previous problem. Let us label the upper left corner as $(0, 6)$ and the bottom right corner as $(9, 0)$. Again the total number of walks between the corners is $\binom{9+6}{6} = \binom{15}{6}$ ways (i.e., we need to choose when to make the 6 down steps among the 15 steps).

We now need to take off the walks using the missing edges. Using the same technique as in the previous problem we have that the number of walks that use the edge $(2, 4) \to (3, 4)$ are the number of walks that first go from $(0, 6)$ to $(2, 4)$ (which can be done in $\binom{2+2}{2} = \binom{4}{2}$ ways), then move to $(3, 4)$ (which can be done in 1 way), then finally move to $(9, 0)$ (which can be done in $\binom{6+4}{4} = \binom{10}{4}$ ways). So there are $\binom{4}{2}\binom{10}{4}$ such walks.

The number of walks that use the edge $(5, 2) \to (5, 1)$ are the number of walks that first go from $(0, 6)$ to $(5, 2)$ (which can be done in $\binom{5+4}{4} = \binom{9}{4}$ ways), then move to $(5, 1)$ (which can be done in 1 way), then finally move to $(9, 0)$ (which can be done in $\binom{4+1}{1} = \binom{5}{1}$ ways). So there are $\binom{9}{4}\binom{5}{1}$ such walks.

But wait! there are some walks that can use both edges so we took off too much. So we need to add back in walks that use both edges. The number of these walks that first go from $(0, 6)$ to $(2, 4)$ (which can be done in $\binom{2+2}{2} = \binom{4}{2}$ ways), then move to $(3, 4)$ (which can be done in 1 way), then move to $(5, 2)$ (which can be done in $\binom{2+2}{2} = \binom{4}{2}$ ways), then move to $(5, 1)$ (which can be done in 1 way), then finally move to $(9, 0)$ which can be done in $\binom{4+1}{1} = \binom{5}{1}$ ways. So there are $\binom{4}{2}\binom{4}{2}\binom{5}{1}$ such walks.

Combining the number of walks is

$$\binom{15}{6} - \binom{4}{2}\binom{10}{4} - \binom{9}{4}\binom{5}{1} + \binom{4}{2}\binom{4}{2}\binom{5}{1} = 3295.$$

3. For $n \geq 0$, find

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}.$$

(Hint: by the general binomial theorem we have $\dfrac{1}{\sqrt{1-4x}} = \sum_{k\geq 0}\binom{2k}{k}x^k$.)

Looking at the hint if we let $a_k = \binom{2k}{k}$ then the sum can be written as

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = \sum_{k=0}^{n} a_k a_{n-k},$$

and in this form we recognize it as the coefficient of x^n when we multiply the generating function by itself. In particular, we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n = \left(\sum_{k=0}^{\infty}\binom{2k}{k}x^k\right)\left(\sum_{k=0}^{\infty}\binom{2k}{k}x^k\right)$$
$$= \frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{k=0}^{\infty} 4^n x^n.$$

Comparing coefficients of x^n we now see that

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

4. In the English language edition of Scrabble there is one letter with 12 tiles (E), two letters with 9 tiles (A,I), one letter with 8 tiles (O), three letters with 6 tiles (N,R,T), four letters with four tiles (D,L,S,U), one letter with three tiles (G), nine letters with 2 tiles (B,C,F,H,M,P,V,W,Y) and five letters with 1 tile (J,K,Q,X,Z).

Find a generating function which counts the number of "words" of length n that can be made using tiles from a Scrabble set. Briefly explain why you chose to use the type of generating function (i.e., ordinary or exponential) that you did.

Since words depend on the order of the letters we will use an *exponential* generating function. We have the following function:

$$g(x)= \underbrace{\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{12}}{12!}\right)}_{\text{E}} \underbrace{\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{9}}{9!}\right)^2}_{\text{A,I}} \underbrace{\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{8}}{8!}\right)}_{\text{O}}$$
$$\times \underbrace{\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{6}}{6!}\right)^3}_{\text{N,R,T}} \underbrace{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)^4}_{\text{D,L,S,U}} \underbrace{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)}_{\text{G}}$$
$$\times \underbrace{\left(1+x+\frac{x^2}{2!}\right)^9}_{\text{B,C,F,H,M,P,V,W,Y}} \underbrace{\left(1+x\right)^5}_{\text{J,K,Q,X,Z}}.$$

5. Let $\left|\begin{matrix} n \\ m \end{matrix}\right|$ denote the number of partitions of n into exactly m parts. Give a proof for the following:

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \left|\begin{matrix} n-1 \\ m-1 \end{matrix}\right| + \left|\begin{matrix} n-m \\ m \end{matrix}\right|$$

We can group partitions of n into m parts into two (disjoint) groups. Those which have a part of size 1 and those which don't.

The number of partitions of n into m parts with a part of size 1 is the same as the number of partitions of $n-1$ into $m-1$ part (i.e., remove the part of size 1, this reduces the number of parts by 1 and the total sum of the parts by 1). In terms of the Ferrer's diagram this corresponds to removing the last row of a single dot. Therefore there are $\left|\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right|$ of these.

The number of partitions of n into m parts with no part of size 1 is the same as the number of partitions of $n-m$ into m parts (i.e., decrease the size of each part by 1, this will not change the number of parts but will decrease the total sum of parts by m). In terms of the Ferrer's diagram this corresponds to removing the first column of dots. There there are $\left|\begin{smallmatrix} n-m \\ m \end{smallmatrix}\right|$ of these.

So by the rule of addition we have

$$\left|\begin{matrix} n \\ m \end{matrix}\right| = \left|\begin{matrix} n-1 \\ m-1 \end{matrix}\right| + \left|\begin{matrix} n-m \\ m \end{matrix}\right|.$$