Midterm I Material

<u>Addition Rule</u>

If the set we are counting can be broken into *disjoint* pieces, then the size of the set it the *sum* of the size of the pieces.

$$S = \bigcup_{i=1}^{m} S_i,\ S_i \cap S_j = \emptyset \text{ if } i \neq j \ \Rightarrow\ |S| = \sum_{i=1}^{m} |S_i|.$$

<u>Multiplication Rule</u>

Suppose we can describe the elements in our set using a procedure with m steps, where at the ith step we have r_i choices available. (Where the *number* of choices is independent of our previous choices.) Then the size of the set is $r_1 r_2 \cdots r_m$.

<u>Probability is counting</u>

Probability is a measurement of how likely an outcome is to occur. If we are dealing with finitely many possible outcomes and each outcome is equally likely (very common assumptions!) then the probability that a certain outcome occurs is the ratio of the outcomes with the desired result divided by the number of all possible outcomes.

The number of ways to arrange r objects from a set of n possible objects is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

The number of ways to choose r objects from a set of n possible objects is

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Given n_1 objects of type 1, n_2 objects of type 2, …, n_k objects of type k and $n = n_1 + n_2 + \cdots + n_k$ then the number of ways to arrange these objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1! n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

This also counts the number of ways to distribute n_1 objects to bin 1, n_2 objects to bin 2, …, n_k objects to bin k where the objects are all distinct.

Sometimes it is easier to count the complement.

The number of ways to distribute n identical objects into k distinguished boxes is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Identities involving binomial coefficients can be proved in various ways. For instance by using the binomial theorem; or by giving a combinatorial argument (i.e., count some object in two different ways and set the two methods for counting equal to each other).

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k},$$

and

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Generating functions can be used to solve problems involving combinations and arrangements. There are two main types of generating functions we have looked at, ordinary and exponential.

Type of generating function	form	used to count
ordinary	$\sum_{n\geq 0} a_n x^n$	combinations
exponential	$\sum_{n\geq 0} a_n \frac{x^n}{n!}$	arrangements

Ordinary generating functions are useful for counting solutions to

$$n_1 + n_2 + \cdots + n_k = n,$$

where the values of n_i are restricted. (If there is no restriction then the number of solutions is $\binom{n+k-1}{k-1}$.) If $n_i \in S_i$ for each i then the (ordinary) generating function where the nth coefficient counts the number of solutions when right hand side is n is given by

$$g(x) = \left(\sum_{\ell \in S_1} x^\ell\right)\left(\sum_{\ell \in S_2} x^\ell\right)\cdots\left(\sum_{\ell \in S_k} x^\ell\right).$$

Exponential generating functions are useful for counting the number of ways to arrange n_1 objects of type 1, n_2 objects of type 2, …, n_k objects of type k where

$$n_1 + \cdots + n_k = n,$$

and the values of n_i are restricted. (An example of this is counting the number of words that can be formed using some restricted set of letters. If there is no restriction then the number of solutions is k^n.) If $n_i \in S_i$ for each i then the (exponential) generating function where the nth coefficient counts the number of solutions when right hand side is n is given by

$$g(x) = \Big(\sum_{\ell \in S_1} \frac{x^\ell}{\ell!}\Big)\Big(\sum_{\ell \in S_2} \frac{x^\ell}{\ell!}\Big)\cdots\Big(\sum_{\ell \in S_k} \frac{x^\ell}{\ell!}\Big).$$

Basics for ordinary generating functions:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1-x^m)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{mk}$$

$$\frac{1-x^{m+1}}{1-x} = 1 + x + x^2 + \cdots + x^m$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad (\text{for } |x| < 1)$$

$$\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{n-1+k}{k} x^k$$

$$\text{If} \quad f(x) = \sum_{n\geq 0} a_n x^n \quad \text{and} \quad g(x) = \sum_{n \geq 0} b_n x^n$$

$$\text{then} \quad f(x)g(x) = \sum_{n \geq 0} c_n x^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Basics for exponential generating functions:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{\ell x} = \sum_{k=0}^{\infty} \ell^k \frac{x^k}{k!}$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\sum_{k \geq n} \frac{x^k}{k!} = e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots - \frac{x^{n-1}}{(n-1)!}$$

$$\text{If} \quad f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} \quad \text{and} \quad g(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

$$\text{then } f(x)g(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!} \text{ where } c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

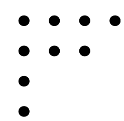
Partitions of n, denoted $p(n)$, are the number of ways to break n up as a sum of smaller terms (the order of the terms is not important). So for instance $p(4) = 5$ since 4 can be partitioned as

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4.$$

The generating function for $p(n)$ is

$$\sum_{n \geq 0} p(n) x^n = \prod_{k \geq 1} \frac{1}{1-x^k}.$$

Partitions can be represented visually as a series of rows of dots. The number of dots in each row is the size of the part and rows are arranged in weakly decreasing order. These are called Ferrer's diagrams. The Ferrer's diagram for the partition $4+3+1+1$ is shown below.



The transpose of the Ferrer's diagram takes rows to columns and columns to rows. The result is a new partition on the same number of points. For example the transpose of 4+3+1+1 is the partition 4+2+2+1. Using transposes it is easy to establish the following fact.

> There is a one-to-one correspondence between the number of partitions of n into exactly m parts and the number of partitions of n into parts of size at most m with at least one part of size m.

It is easy to count partitions with parts of size at most m with at least one part of size m. In particular if we let $\left|{n \atop m}\right|$ denote the number of partitions of n into exactly m parts then we have

$$\sum_{n \geq 0} \left|{n \atop m}\right| x^n = \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}.$$