

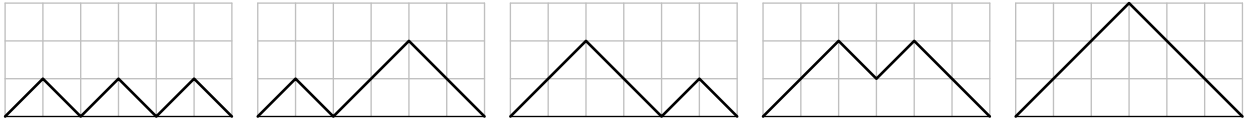
Homework 4 – Due Wednesday, April 29

Section A.2 (page 417): 1 (also give a non-induction proof), 9, 12, 16, 21

Section 7.1 (page 287): 3, 6, 11, 21, 27, 38

Supplemental problems:

1. A Dyck path of length $2n$ is a walk in the plane from $(0, 0)$ to $(2n, 0)$ with steps of the form $(x, y) \rightarrow (x+1, y+1)$ and $(x, y) \rightarrow (x+1, y-1)$ that does not go below the x -axis. Let C_n be the number of Dyck paths of length $2n$. We let $C_0 = 1$. The first few numbers are $C_1 = 1$, $C_2 = 2$ and $C_3 = 5$ (the last case is shown below). (The numbers C_i are called the Catalan numbers and show up in hundreds of different mathematical problems.)



- (a) Show that $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$. (Hint: show that the number of walks that go *below* the x -axis is in one-to-one correspondence with walks that start from $(0, -2)$ and go to $(2n, 0)$.)
- (b) Show that $C_{n+1} = \sum_{k=0}^n C_k C_{n-k}$.
- (c) Show that if $f(x) = \sum_{n=0}^{\infty} C_n x^n$ is the ordinary generating function for the Catalan numbers then the relationship in part (b) implies $f(x) = x(f(x))^2 + 1$.
- (d) Solve the quadratic equation in part (c) to find a simple expression of the ordinary generating function for the Catalan numbers. (Make sure to decide which of \pm to go with in the quadratic formula.)
2. The n th Bell number, $B(n)$, counts the number of different ways to partition the set $\{1, 2, \dots, n\}$. We let $B(0) = 1$. The first few terms are $B(1) = 1$, $B(2) = 2$, $B(3) = 5$ (i.e., $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{\{1, 2, 3\}\}$).
- (a) Show that $B(n+1) = \sum_{k=0}^n \binom{n}{k} B(n-k)$.
- (b) If $g(x) = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$ is the exponential generating function for the Bell numbers, show $g'(x) = e^x g(x)$.
- (c) Solve the differential equation in part (b) to find a simple expression of the exponential generating function for the Bell numbers.
3. Consider a variation of the tower of Hanoi problem where we have three poles A , B and C and start with a stack of n different sized discs on pole A arranged going from the bottom to the top by largest to smallest. We again consider the problem of moving the discs from pole A to pole C . There are three rules: (i) We can only move one disc at a time. (ii) We can never put a larger disc over a smaller disc. (iii) We can not move a disc directly from A to C or from C to A . So for example, if $n = 1$ we now need two moves to transfer from pole A to pole C , first move from A to B and second move from B to C .
- (a) Let $R(n)$ be the minimal number of moves needed to move all n discs from pole A to pole C . Set up a recurrence for $R(n)$, explain your reasoning behind the recurrence.
- (b) Find $R(1)$, $R(2)$, $R(3)$, $R(4)$, and in general find $R(n)$.
4. The Fibonacci numbers F_n are (correctly) defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
- (a) Prove by induction that for $n \geq 1$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.
- (b) Show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. (Hint: use determinants on part (a).)
- (c) Show that $F_{n+1}F_m + F_nF_{m-1} = F_{m+n}$. (Hint: $A^m A^n = A^{m+n}$.)
- (d) Show that F_n divides F_{kn} for $k \geq 1$. (Hint: use induction along with part (c).)