Student name: ____________________________

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MATH 180 (Butler)

Final, 11 June 2009

This test is open book and open notes. Calculator are allowed for this test, but are not needed. For full credit show all of your work and (briefly) explain your approach (legibly!). The score for each problem is as indicated below. On questions 1, 2 and 3 you only have to do one of two options, if you do both we will use whatever one you score higher on.

1	/15
2	/20
3	/20
4	/10
5	/10
6	/15
7	/10
Σ	/100

1. Option 1:

You have 10 cookies and 7 donuts that you are distributing among 5 students.

(a) How many ways are there to distribute the cookies and donuts if there are no restrictions?

Since we treat the cookies as identical and there are 10 cookies and 5 students then there are $\binom{10+5-1}{5-1} = \binom{14}{4}$ ways to distribute the cookies (i.e., using "bars and stars" argument). Similarly there are $\binom{7+5-1}{5-1} = \binom{11}{4}$ ways to distribute the donuts. So by the rule of multiplication there are a total of

$$\binom{14}{4}\binom{11}{4} = 330,330 \text{ ways.}$$

(b) What is the probability that every student gets at least one sweet? (Be careful, we are not requiring that everyone get at least one donut *and* at least one cookie, that is a different question.)

We can apply the principle of inclusion-exclusion. Let A_i be the number of ways so that the ith person does not get a sweet. Then we want to find

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|.$$

Note that the size of the intersection of some of these sets (i.e., $A_2 \cap A_4$) is the number of ways to distribute the sweets among the remaining people (in this case $1, 3, 5$). In general the number of ways to distribute the sweets among k people is

$$\binom{9+k}{10}\binom{6+k}{7}.$$

So we have that the number of ways to do this is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}| = |\mathcal{U}| + \sum_{\substack{I \subseteq [5] \\ I \neq \emptyset}} (-1)^{|I|} \left| \bigcap_{j \in I} A_j \right| = \sum_{k=0}^{5} (-1)^{5-k} \binom{5}{k}\binom{9+k}{10}\binom{6+k}{7} = 181,615.$$

Finally to get the probability we divide by the total number of ways to distribute the sweets giving us

$$\frac{181,615}{330,330} = \frac{5,189}{9,438} = 0.549798\ldots.$$

1. Option 2:

How many arrangements of "MISSISSIPIANS" are there with no "SSS"?

We first group the letters of MISSISSIPIANS by type. We have 1–"M", 1–"N", 1–"P", 1–"A", 4–"I" and 5–"S" (13 letters in all). The total number of ways to arrange these letters is

$$\frac{13!}{1!1!1!1!4!5!} = 2,162,160.$$

The problem of course is that this also counts many configurations with "SSS" so we now need to subtract off. So let us introduce a new letter, namely "SSS" so that we now have 11 letters (1–"M", 1–"N", 1–"P", 1–"A", 4–"I", 2–"S" and 1–"SSS"). Then the total number of ways to arrange these letters is

$$\frac{11!}{1!1!1!1!4!2!1!} = 831,600.$$

The problem is that some arrangements got counted multiple times, for instance words with "SSSS" will be counted twice by this and words with "SSSSS" will be counted three times by this. So we need to correct for this overcounting. To do this we introduce a new letter, namely "SSSS" so that we now have 10 letters (1–"M", 1–"N", 1–"P", 1–"A", 4–"I", 1–"S" and 1–"SSSS"). Then the total number of ways to arrange these letters is

$$\frac{10!}{1!1!1!1!4!1!1!} = 151,200.$$

Note that words with "SSSS" will be counted once by this and words with "SSSSS" will be counted twice by this (so this takes care of both of our overcounting problems) so we have that the total number of rearrangements is

$$2,162,160 - 831,600 + 151,200 = 1,481,760$$

An alternative method is to do the following. First arrange all the non "S" letters, which can be done in

$$\frac{8!}{1!1!1!1!4!} = 1,680 \text{ ways.}$$

We now insert the S's in. There are 9 slots they can go into. We have one of several options for putting the S's in. (1) No repeated S, so each slot gets at most one S which can be done in $\binom{9}{5} = 126$ ways. (2) One slot get's two S's and the other's get at most one which can be done in $\binom{9}{1}\binom{8}{3} = 504$ ways. (3) Two slotss get two S's and the other slot gets one which can be done in $\binom{9}{2}\binom{8}{1} = 252$ ways. So in total the number of rearrangements is

$$\frac{8!}{1!1!1!1!4!}\left(\binom{9}{5} + \binom{9}{1}\binom{8}{3} + \binom{9}{2}\binom{7}{1}\right) = 1,481,760 \text{ ways.}$$

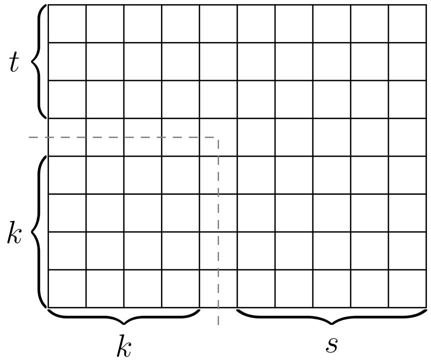
(My apologies to Mississippians who spell their title with two p's, an unintentional typo.)

2. Option 1:

Prove the following identity involving binomial coefficients.

$$\sum_{i=0}^{k}\binom{k+i}{i}\left(\binom{s+t+k+1-i}{s}+\binom{s+t+k+1-i}{t}\right)=\binom{s+t+2k+2}{s+k+1}.$$

(The following picture might be helpful, but you are not required to use it.)



If we label the lower left corner $(0,0)$ and the upper right corner $(s+k+1,t+k+1)$ then we know that the total number of walks between the lower left and the upper right corner using only up steps and right steps are

$$\binom{(s+k+1)+(t+k+1)}{s+k+1}=\binom{s+t+2k+2}{s+k+1},$$

i.e., we have to take $s+t+2k+2$ and we need to choose when among those steps we take a right step.

On the other hand we can break our walk into cases, namely according to the first time we leave the $k\times k$ corner in the lower left. There are two general possibilities

- Go from $(k,i)\to(k+1,i)$ where i is one of $0,1,\ldots,k$. In this case we can get from $(0,0)$ to (k,i) (which can be done in $\binom{k+i}{i}$ ways, i.e., choose the "up" steps) and then go across the edge (in 1 way) and finally go from $(k+1,i)$ to $(s+k+1,t+k+1)$ (which can be done in $\binom{s+t+k+1-i}{s}$ ways, i.e., choose the "right" steps). So for each i this can be done in
$$\binom{k+i}{i}\binom{s+t+k+1-i}{s}\ \text{ways.}$$
- Go fro $(i,k)\to(i,k+1)$ where i is one of $0,1,\ldots,k$. In this case we can get from $(0,0)$ to (i,k) (which can be done in $\binom{k+i}{i}$ ways, i.e., choose the "right" steps) and then go across the edge (in 1 way) and finally go from $(i,k+1)$ to $(s+k+1,t+k+1)$ (which can be done in $\binom{s+t+k+1-i}{t}$ ways, i.e., choose the "up" steps). So for each i this can be done in
$$\binom{k+i}{i}\binom{s+t+k+1-i}{t}\ \text{ways.}$$

Finally, each walk from the lower left to the upper right must use exactly one of these corners so the total number of walks is the sum of the above two terms for $i=0,1,\ldots,k$, which is exactly what we wanted to prove.

2. Option 2:

(a) Show that for $n = 6$ the number of partitions with no part divisible by 3 is equal to the number of partitions with no part repeated 3 or more times by listing the partitions of each type and showing that there are an equal number of each.

Partitions with no part divisible by 3:

$$1+1+1+1+1+1, \quad 1+1+1+1+2, \quad 1+1+2+2, \quad 1+1+4, \quad 1+5, \quad 2+2+2, \quad 2+4$$

Partitions with no part repeated 3 or more times:

$$1+1+2+2, \quad 1+1+4, \quad 1+2+3, \quad 1+5, \quad 2+4, \quad 3+3, \quad 6$$

In particular we see that there are 7 of each.

(b) Give a proof that for all n the number of partitions with no part divisible by 3 is equal to the number of partitions with no part repeated 3 or more times. (You can either give a bijective proof or use generating functions.)

Bijective Proof:

Suppose that we have a partition with no part divisible by 3. Repeat the following until it can no longer be done: any term repeated three (or more) times replace the triple with their sum, i.e., $a + a + a \mapsto 3a$.

This maps partitions with no part divisible by 3 to partitions with no part repeated 3 or more times (i.e., anything repeated three or more times would have been replaced). This is a bijection since we can also go backwards. Repeat the following until it can no longer be done: any term that is a multiple of three break into three equal sized terms, i.e., $3a \mapsto a + a + a$.

Generating Function Proof:

The generating function for no part divisible by 3 is

$$f(x) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}\frac{1}{1-x^5}\frac{1}{1-x^7}\frac{1}{1-x^8}\cdots = \prod_{k\geq 0}\left(\frac{1}{1-x^{3k+1}}\frac{1}{1-x^{3k+2}}\right),$$

i.e., in the denominator all terms of the form $1-x^i$ where i is not a multiple of 3. The generating function for no part repeated three or more times is

$$g(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots = \prod_{k\geq 1}(1+x^k+x^{2k}).$$

Using $(1+u+u^2)(1-u) = 1-u^3$ and lots of cancellation we have

$$g(x) = \prod_{k\geq 1}(1+x^k+x^{2k}) = \prod_{k\geq 1}\frac{1-x^{3k}}{1-x^k} = \frac{\prod_{k\geq 1}(1-x^{3k})}{\prod_{k\geq 1}(1-x^k)} = \prod_{k\geq 0}\left(\frac{1}{1-x^{3k+1}}\frac{1}{1-x^{3k+2}}\right) = f(x).$$

In particular since $f(x) = g(x)$ then we can conclude that what they count is equal, establishing the identity.

3. Option 1:

(a) Given that $f(x) = \sum_{k\geq 0} a_k x^k$ and $\frac{f(x)}{1-x} = \sum_{k\geq 0} b_k x^k$ express the b_k in terms of the a_k.

We recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^2 + \cdots = \sum_{k=0}^{\infty} x^k,$$

and that

$$\left(\sum_{k=0}^{\infty} A_k x^k\right)\left(\sum_{k=0}^{\infty} B_k x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} A_i B_{k-i}\right)x^k$$

to see that

$$\frac{f(x)}{1-x} = \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i \cdot 1\right)x^k.$$

In particular we see that

$$b_k = \sum_{i=0}^{k} a_k.$$

(b) Let F_n denote the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$), and let $p_n = \sum_{k=0}^{n} F_k$. Find a simple expression for the generating function $g(x) = \sum_{k\geq 0} p_k x^k$.

(Hint: the generating function for the Fibonacci numbers is $\sum_{k\geq 0} F_k x^k = \frac{x}{1-x-x^2}$.)

From the previous part we see that the p_n corresponds to the nth coefficient of the generating function for the Fibonacci numbers divided by $1-x$. Since we know the generating function for the Fibonacci numbers we can conclude that

$$g(x) = \sum_{k\geq 0}\left(\sum_{k=0}^{n} F_k\right)x^k = \frac{\sum_{k\geq 0} F_k x^k}{1-x} = \frac{x}{(1-x)(1-x-x^2)}.$$

Option 1: (continued)

(c) Find a generating function $h(x)$ for q_n where $q_0 = 0$, $q_1 = 1$ and $q_n = q_{n-1} + q_{n-2} + 1$ for $n \geq 2$. (Note that $q_n = F_{n+2} - 1$ since it satisfies the initial conditions and the recurrence.)

We have

$$\begin{aligned} h(x) &= \sum_{n\geq 0} q_n x^n \quad = \quad q_0 + q_1 x + \sum_{n\geq 2} q_n x^n \\ &= x + \sum_{n\geq 2}(q_{n-1} + q_{n-2} + 1)x^n \\ &= x + \sum_{n\geq 2} q_{n-1}x^n + \sum_{n\geq 2} q_{n-2}x^n + \sum_{n\geq 2} x^n \\ &= x + x\underbrace{\sum_{n\geq 1} q_{n-1}x^{n-1}}_{=h(x)} + x^2\underbrace{\sum_{n\geq 2} q_{n-2}x^{n-2}}_{=h(x)} + x^2\underbrace{\sum_{n\geq 2} x^{n-2}}_{=1/(1-x)} \\ &= x + xh(x) + x^2h(x) + \frac{x^2}{1-x}. \end{aligned}$$

Rearranging we have

$$(1 - x - x^2)h(x) = x + \frac{x^2}{1-x} = \frac{x}{1-x},$$

or solving for $h(x)$ we have

$$h(x) = \frac{x}{(1-x)(1-x-x^2)}.$$

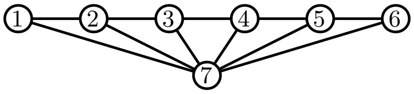
(d) Comparing the answers to (b) and (c) above, what can we conclude about $\sum_{k=0}^{n} F_k$ and $(F_{n+2} - 1)$?

Clearly we have that $g(x) = h(x)$ and so the corresponding coefficients must agree. In particular, we can conclude that

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1.$$

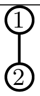
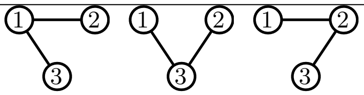
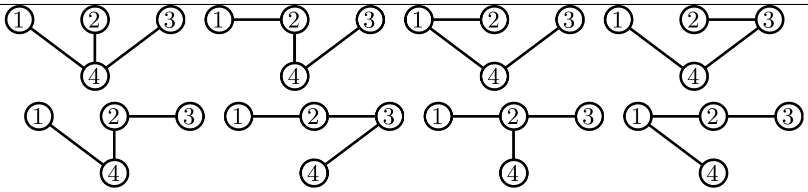
3. Option 2:

A (labelled) n-fan consists of a path on n vertices along with one other vertex that is connected to each vertex on the path, an example is shown below for $n = 6$.



(a) Let t_n be the number of spanning trees of the n-fan. By convention we let $t_0 = 1$ since it consists of a single vertex. Find t_1, t_2 and t_3 by drawing all possible spanning trees for each of the cases.

We have the following.

n	spanning trees	t_n
1		1
2		3
3		8

(b) Show that $t_n = \sum_{i=0}^{n}(n-i)t_i$ for $n \geq 1$.

In a spanning tree of the fan we are breaking the path on the top into pieces (each one a path) and then we are connecting each piece to the vertex $n+1$. So let us break the number of spanning trees of the n-fan according to the size of the last piece (i.e., the piece ending at vertex n). Suppose that this piece has k vertices where k can be one of $1, 2 \ldots, n$. Then we have to pick one of these k vertices to the vertex $n+1$ (which can be done in k ways), the remainder of the spanning tree comes from a spanning tree on the $(n-k)$-fan and so we pick one of these in t_{n-k} ways. So there are kt_{n-k} ways that the last piece has k vertices. We now sum so we have that

$$t_n = \sum_{k=1}^{n} kt_{n-k} = \sum_{i=0}^{n-1}(n-i)t_i = \sum_{i=0}^{n}(n-i)t_i.$$

In the middle stepped we just changed the index $k \mapsto n - i$ and in the last step we note that when $i = n$ the term inside is 0.

Option 2: (continued)

(c) Find a simple expression for the generating function $T(x) = \sum_{k\geq 0} t_k x^k$.

Recalling (as we did in the other option for this problem) that

$$\left(\sum_{k=0}^{\infty} A_k x^k\right)\left(\sum_{k=0}^{\infty} B_k x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} A_i B_{k-i}\right)x^k,$$

we see that the recursion from part (b) implies that the generating function satisfies the following

$$\begin{aligned} T(x) &= \sum_{k\geq 0} t_k x^k = 1 + \sum_{k\geq 1} t_k x^k \\ &= 1 + \sum_{k\geq 1}\left(\sum_{i=0}^{k}(k-i)t_i\right)x_k \\ &= 1 + \Bigg(\underbrace{\sum_{k\geq 0} t_k x^k}_{=T(x)}\Bigg)\left(\sum_{k\geq 0} kx^k\right). \end{aligned}$$

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

if we differentiate both sides and then multiply through by x we have

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k.$$

So we have that

$$T(x) = 1 + \frac{x}{(1-x)^2}T(x),$$

or rearranging

$$1 = T(x) - \frac{x}{(1-x)^2}T(x) = \frac{1-3x+x^2}{(1-x)^2}T(x).$$

Finally, solving for $T(x)$ we can conclude that

$$T(x) = \frac{1-2x+x^2}{1-3x+x^2}.$$

(On a fun side note the numbers that come out of this generating function are every other Fibonacci number.)

4. Recall that for a connected graph G, a bridge is an edge whose removal disconnects the graph. Given that G is a k-regular bipartite graph with $k \geq 2$ show that G has no bridge. (Be careful, k can be even or odd!)

Suppose that G was a k-regular bipartite graph and that it did have a bridge. Then we can remove the bridge and that splits the graph into two pieces, let us look at one of these pieces. This graph is bipartite so let us label the two sets of vertices as U and W (all edges go between U and W. Let us suppose that the bridge was adjacent to an edge in U, then we have one vertex of degree $k-1$ and all the other vertices of degree k, in particular we have that the sum of the degrees in U is of the form $|U|k-1$. On the other hand all the vertices in W have degree k and so now the sum of the degrees in V is $|V|k$.

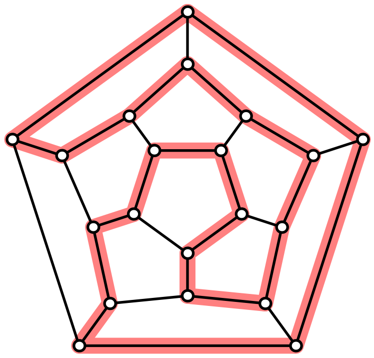
Finally we make an observation, if we add up the degrees in U we count each edge exactly once, similarly for the sum of the degrees in W. So we now have

$$|U|k-1=\sum_{v\in U} d(v) = (\#\text{ of edges}) = \sum_{v\in W} d(v) = |V|k.$$

But for $k \geq 2$ this last statement is impossible (i.e., it says that $(|U|-|V|)k=1$ but no integer $k \geq 2$ divides 1). So there must not be a bridge.

5. Hamiltonian graphs are related to the icosian game created by William Hamilton that involved finding a way to visit all of the vertices of a dodecahedron once and return to the starting vertex. The graph of the dodecahedron is shown below, "win" the game by marking a hamiltonian cycle in the graph.

One hamiltonian cycle is shown below (there are of course many others).



6. The *independence number* of a graph G, denoted $\alpha(G)$, is the maximum number of vertices that can be chosen so that no two are adjacent. For example $\alpha(K_n) = 1$ (since any two are adjacent we can only choose one) and $\alpha(C_5) = 2$ (since if we pick any three vertices on the five cycle two would have to be adjacent to one another).

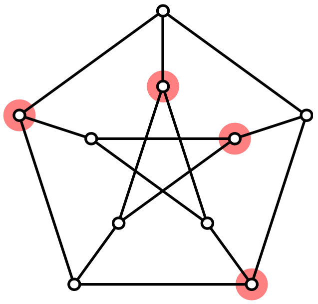
(a) Show that if a graph has n vertices then $n \le \chi(G)\alpha(G)$. (Recall that $\chi(G)$ is the chromatic number of a graph.)

When we have a proper coloring of a graph the set of vertices of one fixed color are independent (i.e., do not have an edge in common). So suppose that $\chi(G) = k$ then we color the graph and split the vertices up according to the colors, i.e., we let V_1 be the vertices colored with the first color, V_2 be the vertices colored with the second color, and so on. By definition we have that $|V_i| \le \alpha(G)$ for each i since each V_i is a set of independent vertices. So we have

$$n = |V_1| + |V_2| + \cdots + |V_k| \le \underbrace{\alpha(G) + \alpha(G) + \cdots + \alpha(G)}_{k=\chi(G)\text{ terms}} = \chi(G)\alpha(G).$$

(b) Find $\alpha(P)$ where P is the Petersen graph. (You should have two parts to your answer, find a set with $\alpha(P)$ nonadjacent vertices and explain why no set with $\alpha(P) + 1$ vertices can be all mutually nonadjacent.)

It is not too hard to find an independent set of size 4 for the Petersen graph, one such example is shown below.



We now show that we cannot have an independent set of size 5. To see this note that we can think of the Petersen graph as an outer five cycle an inner five cycle (the star) and edges in between. Since on each five cycle we can have at most two independent vertices (since $\alpha(C_5) = 1$) then for our independent set in P we get at most two vertices from the outer five cycle and two from the inner five cycle and this exhausts since this covers all the vertices there is nowhere else for us to get a fifth independent vertex. So we can conclude that $\alpha(P) = 4$.

7. Three interesting things I learned from taking this course are . . .
(There are no incorrect answers, as long as you write something you will receive credit.)

Answers vary, but certainly one thing that we have hopefully learned by now is that when Professor Butler says that a test is “easy”, he is lying to you!