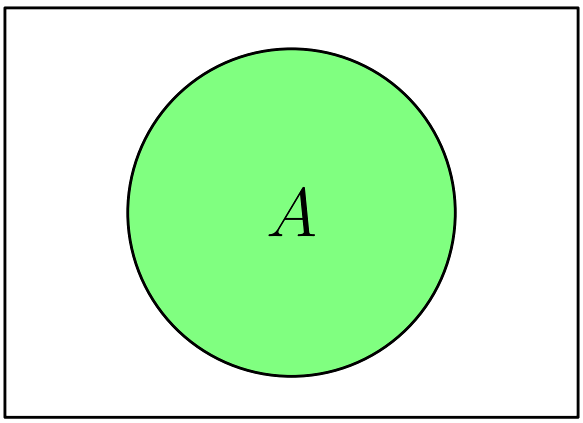
Lecture 16 – May 6

We now return to some good old fashioned counting. We will be looking at sets. We have a collection of elements and we will let $\mathcal{U}$ denote the "univers" of all possible elements for the problem that we are counting. We will let A denote a subset of $\mathcal{U}$, $|A|$ denote the number of elements in the set A, and $\overline{A} = \mathcal{U} \setminus A$ denote the complement of A (the set of all elements in $\mathcal{U}$ not in A). Since every element is either in A or not in A we can conclude

$$|\mathcal{U}| = |A| + |\overline{A}|.$$

Pictorially we have the following picture.



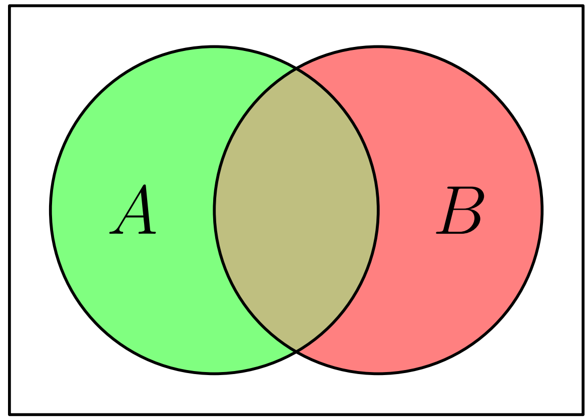
This is a simple example of a Venn diagram which gives a visual way to represent sets. We will look at Venn diagrams for 1, 2 and 3 sets, after that it gets more difficult to draw Venn diagrams (but not impossible!).

Example: There are 35 students in the class. If 15 students come to office hours then how many student's didn't come.

Solution: The set $\mathcal{U}$ is the set of all students, A the set of students who came to office hours so by the above formula we have

$$|\overline{A}| = |\mathcal{U}| - |A| = 35 - 15 = 20.$$

Not surprisingly, there is not much interesting to do with only one set. So now let us consider the situation when we have two set A and B. The Venn diagram for this situation is shown below.



Notice that this picture implies the possibility that the sets intersect. We do this because we ant Venn diagrams to be as general as possible, so the actual intersection can be empty (do not assume that sets always intersect). We let $A \cup B$ denote the union of A and B where

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\},$$

and $A \cap B$ denote the intersection of A and B where

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}.$$

de Morgan's laws

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} \\ \overline{A \cup B} &= \overline{A} \cap \overline{B} \end{aligned}$$

In words de Morgan's laws says that the complement of the union is the intersection of the complements, and that the complement of the intersection is the union of the complements. To prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$ we show that they have the same elements.

$$\begin{aligned} x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B \\ &\Leftrightarrow x \notin A \text{ or } x \notin B \\ &\Leftrightarrow x \in \overline{A} \text{ or } x \in \overline{B} \\ &\Leftrightarrow x \in \overline{A} \cup \overline{B} \end{aligned}$$

The other half of the law is proved similarly.

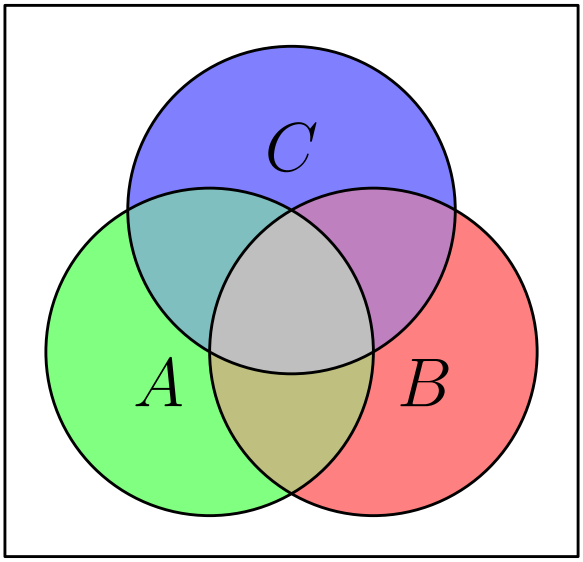
Suppose that we wanted to count the number of elements in $|A \cup B|$, a natural first guess is $|A| + |B|$. The problem is that we are double counting elements in the intersection of A and B, so we need to correct this overcounting by subtracting out. So we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can also count this by breaking $A \cup B$ into three pieces, namely $A \cap \overline{B}$ (elements in A but not in B), $A \cap B$ (elements in A and in B) and $\overline{A} \cap B$ (elements not in A but in B). So we also have

$$|A \cup B| = |A \cap \overline{B}| + |A \cap B| + \overline{A} \cap B|.$$

We now turn to the three set case. This is illustrated in the Venn diagram below.



Notice that we have split $\mathcal{U}$ into eight pieces, which is what we should expect because we are either in or not in each of the three sets. The eight pieces are

$$A\cap B\cap C,\ A\cap B\cap \overline{C},\ A\cap \overline{B}\cap C,\ A\cap \overline{B}\cap \overline{C},$$
$$\overline{A}\cap B\cap C,\ \overline{A}\cap B\cap \overline{C},\ \overline{A}\cap \overline{B}\cap C,\ \overline{A}\cap \overline{B}\cap \overline{C}.$$

The last set, by de Morgan's law is $\overline{A}\cap\overline{B}\cap\overline{C} = \overline{A\cup B\cup C}$ which are the points outside of the sets A, B and C.

Similarly as with the two set case we may want to count the union of the sets, and as before we have to be careful not to overcount. So we have

$$|A\cup B\cup C| = |A|+|B|+|C| \\ -|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|.$$

Example: At a local college there is a language program which has the three languages Russian (R), Chinese (C) and Java (J). There are 34 students attending the Russian class, 31 students taking the Chinese class, 36 students taking the Java class, 13 students taking both Russian and Chinese, 9 students taking Russian and Java and 11 students taking Chinese and Java. When this is reported to the dean they look at it and say, "Wait, I also need to know how many students are in all three!" How many students attend all three classes?

Solution: Thinking of these as sets then we are looking for $|R\cap C\cap J|$. Plugging in the information we have into the formula above we have

$$73 = 34+31+36-13-9-11+|R\cap C\cap J| \\ = 68+|R\cap C\cap J|.$$

So $|R\cap C\cap J| = 5$, i.e., there are 5 students taking all three classes.

Example: How many n digit ternary sequences with at least one 0, one 1 and one 2?

Solution: Let us use sets to represent the situation. Let A denote the set of sequences without 0, B denote the sequences without 1 and C denote the sequences without 2. (We could alternatively let them denote the sets *with* the elements, but this is more fun.)

Phrased this way we are looking for

$$|\overline{A}\cap\overline{B}\cap\overline{C}| = |\overline{A\cup B\cup C}| = |\mathcal{U}| - |A\cup B\cup C|.$$

On the one hand we have $|A| = |B| = |C| = 2^n$, since elements in these sets are formed only using two letters. We also have $|A\cap B| = |A\cap C| = |B\cap B| = 1$ since elements in these sets are formed only using one letter (so they look like $000\cdots0$, $111\cdots1$ and $222\cdots2$). Finally we have $|A\cap B\cap C| = 0$ since we don't have any letters to form the sequence. Putting this altogether we have

$$|\overline{A}\cap\overline{B}\cap\overline{C}| = 3^n - \big(3{\cdot}2^n - 3{\cdot}1 + 0\big) = 3^n - 3{\cdot}2^n + 3.$$

It is always a good idea to check, so let us try 0, plugging it in we get 1, but that is not the right answer, we should have gotten 0 (there are no sequences of length 0 with at least one 0, one 1 and one 2). So where was our mistake? Looking back over what we did we see that $|A\cap B\cap C|$ will be 1 when $n = 0$ and 0 otherwise. This is because the empty string (the string with no letters) has no 0, no 1 and no 2 and so is an element in $|A\cap B\cap C|$. So if we put this back in we see that we do get the correct answer. So we have that the number of such ternary sequences is

$$\begin{cases} 3^n - 3{\cdot}2^n + 3 & \text{if } n \ge 1, \\ 0 & \text{if } n = 0. \end{cases}$$

What we have just done is a simple example of a more basic principle.

> *Inclusion-Exclusion Principle:*
> Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then
> $$\big|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\big| = |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}} (-1)^{|I|}\Big|\bigcap_{j\in I} A_j\Big|.$$

Before we try to prove this we first should work on trying to understand what this says. We have $[n] = \{1, 2, \ldots, n\}$, so what we are doing is summing over all nonempty subsets of $[n]$. What this is really doing is summing over all possible ways that we can intersect the sets A_i. Where we either add or subtract based on how many sets we are intersecting. For example for $n=3$ this corresponds to the following,

$$
\begin{aligned}
|\overline{A_1}\cap\overline{A_2}\cap\overline{A_3}| = |\mathcal{U}| - \underbrace{|A_1|}_{I=\{1\}} - \underbrace{|A_2|}_{I=\{2\}} - \underbrace{|A_3|}_{I=\{3\}} \\
+ \underbrace{|A_1\cap A_2|}_{I=\{1,2\}} + \underbrace{|A_1\cap A_3|}_{I=\{1,3\}} + \underbrace{|A_2\cap A_3|}_{I=\{2,3\}} \\
- \underbrace{|A_1\cap A_2\cap A_3|}_{I=\{1,2,3\}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
|A_1\cup A_2\cup\cdots\cup A_n| &= |\mathcal{U}| - |\overline{A_1\cup A_2\cup\cdots\cup A_n}| \\
&= |\mathcal{U}| - |\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|
\end{aligned}
$$

we automatically have the following variation of the inclusion-exclusion principle.

> *Inclusion-Exclusion Principle II:*
> Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then
>
> $$\big|A_1\cup A_2\cup\cdots\cup A_n\big| = \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}} (-1)^{|I|-1}\Big|\bigcap_{j\in I} A_j\Big|.$$

The proof of the inclusion-exclusion principle is based on a binomial identity that we encountered earlier, namely

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k} = \begin{cases} 1 & \text{if } m=0,\\ 0 & \text{if } m\geq 1.\end{cases}$$

The way to do this is to go through every element in $\mathcal{U}$ and see how many times it gets counted on each side of the equation. We have two cases to consider.

- $x\in|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|$, note that x is in *none* of the sets A_i. On the left hand side this gets counted and so contributes 1. On the right hand side this gets counted when we look at the term $|\mathcal{U}|$ and since it is in none of the A_i it will not get counted any other time so the total contribution is 1.

- $x\notin|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|$, now x is in some of the sets A_i, let us suppose it is in m sets. On the left hand side this does not get counted and so contributes 0. On the right hand side this will get counted many times, once for the $|\mathcal{U}|$, then $\binom{m}{1}$ times for each subset it is in, then $\binom{m}{2}$ times for each pair of subsets it is in, then $\binom{m}{3}$ times for each triple of subsets it is in, and so on. In particular, the contribution on the right hand side (using signs) is
$$1-\binom{m}{1}+\binom{m}{2}-\binom{m}{3}+\cdots=0.$$

In both cases the left and right sides are equal for each element establishing the result.

Lecture 17 – May 8

We now look at some applications of how to use the inclusion-exclusion principle. A classic example is counting *derangement*. A derangement can be thought of as a permutation (a function $\pi:[n]\to[n]$) with no fixed point (an i so that $\pi(i)=i$). This is often rephrased as a hat-check problem, where n people go out to eat and check in their hats but when they leave the person in charge of giving back their hats has forgotten who goes with which hat and so gives them back randomly, a derangement then corresponds to the situation where no person gets their own hat back.

Example: How many derangement on n elements are there?

Solution: To use the inclusion-exclusion principle we want to first identify the sets A_i. There are many possibilities but we keep in mind two criteria. First, we want to find the intersection of the complement of the A_i. Second, we want to find sets which are easy to count when we look at their intersections. Given these two criteria we are led to

$$A_i = \{i\text{th person gets their hat}\},$$

so that $\overline{A_i}$ denotes arrangements where A_i does not get their hat back. Since no one should get their hat back we want to look at the intersection of these sets. So the number of derangements is

$$\big|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\big| = |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}}(-1)^{|I|}\Big|\bigcap_{j\in I}A_j\Big|.$$

Going through these terms we have $|\mathcal{U}|$ is the number of ways to distribute the hats which is $n!$. Now let us consider a term of the form $\cap_{j\in I}A_j$. This corresponds to arrangements that are simultaneously satisfying that elements in I get their hat back. The remaining $n-j$ can be distributed arbitrarily. So we have

$$\Big|\bigcap_{j\in I}A_j\Big| = (n-|I|)!.$$

(Importantly, we see that the only thing that is important is the number of elements of I and not which elements they are.) If we now group the sum by the size of the index set we have

$$
\begin{aligned}
\big|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\big| &= |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}}(-1)^{|I|}\Big|\bigcap_{j\in I}A_j\Big| \\
&= n! + \sum_{k=1}^{n}(-1)^k\binom{n}{k}(n-k)! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.
\end{aligned}
$$

Then term $\binom{n}{k}$ shows up because there are $\binom{n}{k}$ ways to choose k out of n sets to intersect.

Note that the sum that shows up in the number of derangements looks familiar. In particular since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{we have} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}.$$

So the number of derangements is $\approx n!/e$. In fact it can be shown that the number of derangements is the integer nearest to $n!/e$.

Using this we can conclude that the probability that if we collect hats from n people and redistribute them randomly that no one gets their hat back is $\approx 1/e$.

Example: Suppose that there are n couples that go to a theatre and sit in a row with $2n$ chairs.

(a) How many ways can the couples sit down if each person sits next to the person they came with?

(b) How many ways can the couples sit down if no person sits next to the person they came with?

Solution: For part (a) we think of pairing the couples together and then arrange the couples (which can be done in $n!$ ways). Now each couple sits down and we need to decide who sits on the left and who sits on the right which can be done in 2 ways per couple, so in total there are $2^n n!$ ways for the couples to sit down.

For part (b) we use inclusion-exclusion. If we let

$$A_i = \{\text{seatings with } i\text{th couple together}\},$$

then what we are looking for is $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$ (just right for inclusion-exclusion). The total number of ways to seat the $2n$ people is $(2n)!$. Now let us focus on a term of the from $\cap_{j\in I} A_j$. This corresponds to the situation where the couples in I are sitting together (and maybe some more as well). To count this we now think of there as being $|I|$ couples and $2n-2|I|$ singles for us to arrange, or combined $2n-|I|$ "people" to arrange. Once we are done arranging the order of the people we then again have to decide for each couple who sits on the left and who sits on the right. So altogether we have that

$$\left| \bigcap_{j\in I} A_j \right| = (2n-|I|)!2^{|I|}.$$

(Again it is important that it is the number of elements in I and not which elements in I that matters.) So as in the last example if we group by the size of the index set we have

$$\begin{aligned} \left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| &= |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}} (-1)^{|I|} \left| \bigcap_{j\in I} A_j \right| \\ &= (2n)! + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (2n-k)!2^k \\ &= \sum_{k=0}^{n} (-2)^k \binom{n}{k} (2n-k)!. \end{aligned}$$

Again the term $\binom{n}{k}$ shows up because there are $\binom{n}{k}$ ways to choose k of the n sets to intersect. Also we can absorb the $(2n)!$ term into the sum since this corresponds to the case $k = 0$.

Unfortunately, there does not seem to be any nice way to condense this formula, but at least we have a way to count it! Using a computer we see that

$$\underbrace{0}_{n=1}, \underbrace{8}_{n=2}, \underbrace{240}_{n=3}, \underbrace{13824}_{n=4}, \underbrace{1263360}_{n=5}, \underbrace{168422400}_{n=6}, \ldots.$$

Finally, let us look at some more variations of the inclusion-exclusion principle

Variation of the Inclusion-Exclusion Principle: Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$ and let

$$B_m = \{x \mid x \text{ in } \textit{exactly } m \text{ of the } A_i\}.$$

Then

$$|B_m| = \sum_{\substack{I\subseteq[n]\\ |I|\geq m}} (-1)^{|I|-m} \binom{|I|}{m} \left| \bigcap_{j\in I} A_j \right|.$$

The case B_0 (where we translate the empty intersection as the entire $\mathcal{U}$) gives us back the original form of the inclusion-exclusion principle. Our method to prove this is similar as before, namely to consider cases for individual elements in $\mathcal{U}$ and show that each elements contributes the same to both sides.

- x in fewer than m of the A_i. In this case $x \notin B_m$ so the contribution on the left is 0 while on the right it will not show up in any of the intersections and so its contribution is again 0.
- x in exactly m of the A_i. In this case $x \in B_m$ so the contribution on the left is 1 while on the right it will show up exactly once, namely in the intersection of the m sets it lies in. It cannot show up in any other term, so the contribution on the right is 1.
- x in $r > m$ of the A_i. In this case $x \notin B_m$ so the contribution on the left is 0. While on the right it will show up $\binom{r}{m}$ times in the intersection of m

sets, $\binom{r}{m+1}$ times in the intersection of $m+1$ sets, $\binom{r}{m+2}$ times in the intersection of $m+2$ sets, ... In particular the contribution on the right hand side will be

$$\binom{r}{m}\binom{m}{m} - \binom{r}{m+1}\binom{m+1}{m} + \binom{r}{m+2}\binom{m+2}{m} - \cdots.$$

Digging back to when we practiced identities for binomial coefficients we find the identity

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

So applying this to our situation we have

$$\binom{r}{m+i}\binom{m+i}{m} = \binom{r}{m}\binom{r-m}{i}.$$

In particular we can rewrite our sum where we use this identity on every term and factor out the (now common) $\binom{r}{m}$ to get that the contribution on the right is

$$\binom{r}{m}\left(\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots\right)$$

which, by the same reasoning as last time, is 0.

Closely related to this we have the following.

> *Variation of the Inclusion-Exclusion Principle (II):* Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$ and let
>
> $$C_m = \{x \mid x \text{ in } \textit{at least } m \text{ of the } A_i\}.$$
>
> Then
>
> $$|C_m| = \sum_{\substack{I\subseteq[n]\\|I|\geq m}} (-1)^{|I|-m}\binom{|I|-1}{m-1}\left|\bigcap_{j\in I} A_j\right|.$$

The case $m = 1$ corresponds to the variation that we gave in the last lecture. We will not give the proof here.

This brings us to the end of the enumerative combinatorics part of the course. Starting with the next lecture we turn to graph theory.

Lecture 18 – May 11

We now start a new area of combinatorics, namely graph theory. Graphs are at a basic level a set of objects and connections between them. As such they can be used to model all sorts of real world and mathematical objects. We start with some of the basic properties of graphs.

A *graph* G consists of two sets, a vertex set V and an edge set E, we sometimes will write this as $G = (V, E)$. The vertex set represents our objects and the edge set represents our connections between objects, we will visually denote vertices by "o". The edge set can have several different ways of describing how we connect vertices which leads to different types of graphs. The most common are:

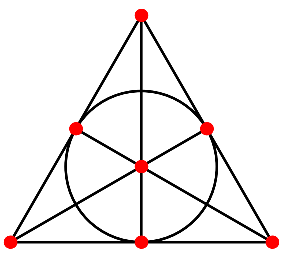
- The set E consists of unordered pairs of vertices, i.e., $\{u, v\}$. These graphs are the most studied and are known as *undirected graphs.* Visually we represent the edges as "o—o".
- The set E consists of an ordered list of two vertices, i.e., (u, v). These graphs are known as *directed graphs.* Visually we represent the edges as "o→o".
- The set E consists of subsets of vertices, usually of some fixed size, i.e., $\{v_1, v_2, \ldots, v_k\}$. These graphs are known as *hypergraphs.* These are hard to represent visually (perhaps this is one of the reasons that we do not study them as much in beginning combinatoric courses).

Usually when we think of a graph we do not think of two sets but rather as a visual drawing of points and lines connecting the points. This is great for building intuition, but can also be deceiving in that the same graph can be drawn in many different ways (we will look at this problem shortly). Another problem is that many graphs that are interesting have a *large* number of vertices (in the billions) so that if we were to draw them as points and lines the end result would be a black piece of paper. So it is good to be able to not have to rely on pictures!

An example of an undirected graph is a handshake graph which corresponds to a party with several people present and we connect an edge between two people if they shake hands. As a graph the vertices are the people and the edges correspond to handshakes. Because there is symmetry in the relationship (i.e., if I shake your hand you also shake mine) the undirected graph is the appropriate one to use. These also can be used to model social networks, in addition they are frequently used to model groups (these graphs are known as Cayley graphs).

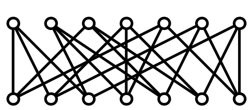
An example of a directed graph is the internet, sometimes known as the web graph. Here every vertex corresponds to a webpage and edges are hyperlinks from one webpage to another webpage. Because there does not have to be symmetry (i.e., I might link to a website about kittens but that does not mean the kitten website links to me) the directed graph is the appropriate one to use. The web graph is one of the most studied (and probably one of the most lucrative) graphs. Companies such as `Google` make a lot of money mining information from this graph.

An example of a hypergraph is the Fano plane. This is the smallest discrete geometry it consists of seven points, seven lines, where each line contains exactly three points, any two lines intersect in exactly one point and any two points are contained in one unique line. This is shown below.

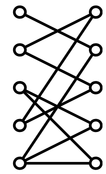


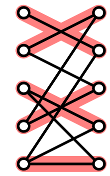
We can represent the Fano plane as a hypergraph by letting the points of the Fano plane be the vertices and the edges of the hypergraphs be the lines. So in this case every edge consists of three points.

One reason that we do not study hypergraphs is that we can model them using undirected bipartite graphs. A graph is *bipartite* if the vertices can be split into two parts $V = U \cup W$ where U and W are disjoint and such that all the edges in the graph connect a vertex in U to a vertex in W. A hypergraph $G = (V, E)$ can then be associated with a bipartite graph $H = (V \cup E, \mathcal{E})$ where $\mathcal{E}$ connects $v \in V$ to $e \in E$ if and only if the vertex v is in the edge e. So for example the hypergraph for the Fano plane can be represented using the following bipartite graph, where points are on the top lines are on the bottom. (While not as nice a picture as the Fano plane it still has all of the same information and we can still use it to determine properties of the Fano plane.)



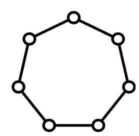
Because of their ability to model interaction between two groups, bipartite graphs are an important area of study in graph theory. As another example suppose that we want to model the social interaction between a group of five boys and five girls. There have been various relationships formed over the years and some people get along great and have fun dating while other couples don't work at all. We can model this by a bipartite graph where on the left we have the girls on the right we have the boys and we put an edge between two people if they can date. Suppose that doing this we get the graph shown below on the left.





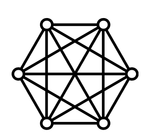
We might then ask the question, is it possible for all the people to have a date for this Friday? In this case we need to find a way to pair people up so that every person is in exactly one pairing and they get along with the person they are paired with. So looking at the graph we see that the first girl would have to be paired with the second boy, the second and fourth girls would then decide how to go with the first and third boys while the third and fifth girls would then decide how to go with the fourth and fifth boys. In particular, there are four ways that people can be paired up for dates, we show one of them above on the right.

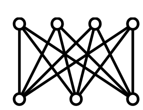
This is an example of a *matching*. A matching of a graph is a subset of edges so that each vertex is in exactly one edge. Note that this definition does not require the graph is bipartite. In the graph above there is a matching, but not all graphs have matchings. For example consider the following graph.



It is easy to see that there is no way to match up the vertices in this graph since there is an *odd* number of vertices (if there were a matching then the number of vertices would have to be a multiple of two). This graph is a special type of graph known as a *cycle*. A cycle on n vertices, denoted C_n, consists of the vertex set $\{1, 2, \ldots, n\}$ with edges $\{i, i+1\}$ for $i = 1, \ldots, n-1$ and $\{1, n\}$. If we do not include the last edge (the $\{1, n\}$) then the graph is a *path* on n vertices, denoted P_n.

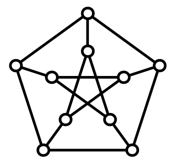
Another well studied graph is the *complete* graph K_n which consists of the vertex set $\{1, 2, \ldots, n\}$ and has all possible edges. Since edges consist of subsets of size two then there are $\binom{n}{2}$ edges in the complete graph. The graph K_6 is shown below on the left.



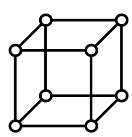


Closely related to the complete graph is the *complete bipartite* graph $K_{m,n}$ which has as its vertex set $\{v_1, \ldots, v_m, u_1, \ldots, u_n\}$ and as edges all edges of the form $\{v_i, u_j\}$. In other word we have a bipartite graph with one part of size m, one part of size n and all possible edges going between the two parts, so in total mn edges. The graph $K_{3,4}$ is shown above on the right.

One of the most famous graphs, the Fabio of graph theory if you will, is the Petersen graph. This graph consists of 10 vertices and 15 edges as shown below. This graph is a classic example/counterexample to many problems in graph theory and is a good graph to have around when testing a new idea, we will use this as an example in a later lecture.



Another important graphs are *hypercubes* Q_n (here the "hyper" refers to high dimensions, these graphs are not themselves hypergraphs). The vertex set of these graphs consist of all binary strings of length n, so that there are 2^n vertices. Two vertices are connected if the strings differ in *exactly* one entry. So for example Q_1 consists of vertices 0 and 1 with an edge connecting them, Q_2 consists of the four vertices 00, 01, 10 and 11 and form a four cycle (note that there will be no edge connecting 00 and 11 since they differ in both entries and similarly no edge connecting 01 and 10). The graph Q_3 is shown below (you should try to label the vertices with the eight binary strings of length three that would give this graph). The name hypercube comes from the idea of the vertices denoting the corners of an n-dimensional cube sitting at the origin, because binary strings are used in computers these graphs frequently arise in computer science.



Note that the hypercube is a bipartite graph. To see this we need to tell how to break up the vertices into two sets so that edges go only between these sets. This is done by letting U be the set of binary strings of length n with an *odd* number of 1s and W the set of binary strings of length n with an *even* number of 1s. Since an edge connects two vertices which differ by exactly one entry we will either increase or decrease the number of 1s by one, so that edges must go between U and W.

Let us now look at how many edges are in the hypercube. Before we do that we first need a few definitions. Two vertices u and v are *adjacent* if there is an edge connecting them, this is denoted as $u \sim v$. An edge e and a vertex v are *incident* if the vertex v is contained in the edge e (pictorially the edge connects to the vertex). The *degree* of a vertex is the number of edges incident to the vertex, equivalently the number of vertices adjacent to the vertex (assuming we do not allow loops and multiple edges), we denote the degree of a vertex by d_v or $d(v)$.

In the hypercube graph Q_n every vertex corresponds to a string of length n. To connect to this vertex we must differ in exactly one letter, since there are n possible letters then there are n different vertices that connect to the graph. In other words every vertex in the graph has degree n. If we add up all the degrees then we get $n2^n$. But adding up all the degrees we will count each edge exactly twice, so that twice the number of edges is $n2^n$ or the number of edges is $n2^{n-1}$. As an example in Q_3 there are $3 \cdot 2^2 = 12$ edges.

A graph where all of the vertices have the same degree is known as a *regular* graph. The Petersen graph is a regular graph of degree 3, the hypercube is a regular graph of degree n, the complete graph K_n is a regular graph of degree $n-1$ and the cycle C_n is regular of degree 2. Of course there are many others, regular graphs have nice properties, particularly when approaching graphs using linear algebra.

Looking at our derivation for the number of edges in Q_n we noted that by adding up the sum of the degrees that we got twice the number of edges. This is the first theorem of graph theory.

Handshake Theorem:
For a graph G let $|E|$ denote the number of edges, then

$$\sum_{v \in V} d(v) = 2|E|.$$

In particular this implies that the sum of the degrees must be an even number. So we have the following immediate corollary.

Corollary of the Handshake Theorem:
The number of vertices with odd degree must be even.

Example: Does there exist a graph on six vertices where the vertices have degree $1, 2, 3, 3, 4, 4$?

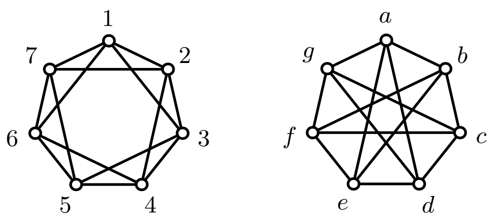
Solution: No. Summing up the degrees we get 17 which is an odd number, but by the handshake theorem if such a graph existed the sum would have to be even.

Of course, just because the sum of the degrees of a graph is even does not mean that a graph exists.

Example: Does there exist a graph on six vertices where the vertices have degree $1, 2, 3, 4, 5, 5$?

Solution: No. Summing up the degrees we get 20 so the handshake theorem does not rule it out. However if there were such a graph then it would have two vertices of degree 5, in particular these two vertices would have to be adjacent to every vertex in the graph. Or put another way, every vertex in the graph would have to be adjacent to the two vertices of degree 5, but this would imply that no vertex can have degree less than 2, in particular we could not have a degree 1 vertex as desired.

Finally, we turn to a problem mentioned earlier. Namely, we like to represent graphs visually but the way that we can draw the graph is not unique as we can place vertices where we like and draw edges as we want. For example consider the following two graphs.



These graphs have the same number of vertices (7) the same number of edges (14) all the degrees are the same, but are they two different drawings of the same graph or two different graphs altogether?

To answer this question we have to decide what we mean to say that they are the same graph, or rather they have the same structure. We say two graphs G and H are *isomorphic* ("iso"=same, "morphic"=structure) if there is a bijective (one-to-one and onto) map $\phi : V(G) \to V(H)$ so that $u{\sim}v$ in G if and only if $\phi(u){\sim}\phi(v)$ in H. Or in other words we can map the vertices of one graph to the other in a way that preserves adjacency.

In terms of the picture above the question is can we relabel the vertices of one graph and produce the other graph. Careful checking will see that if we relabel as follows $1 \mapsto a$, $2 \mapsto c$, $3 \mapsto e$, $4 \mapsto g$, $5 \mapsto b$, $6 \mapsto d$ and $7 \mapsto f$, then we preserve adjacency. In other words these two graphs are isomorphic so are two drawings of the same graph.

To show that two graphs are isomorphic then it suffices for us to find a way to relabel the vertices of one graph to produce the other graph. To find a rule for relabeling we often try to identify some vertices that must be mapped one to the other (i.e., because of degree considerations) and slowly build up the relationship.

To show that two graphs are not isomorphic we need to find some structure that one graph has that the other does not, showing they cannot be the same graph. We will pick this topic up again in the next lecture.

Lecture 19 – May 13

Our main focus in graph theory will be simple undirected graphs. Simple graphs refer to graphs without loops or multiple edges. A loop is an edge that connects a vertex to itself, i.e.,  . A multiple edge is several edges going between the same two vertices, i.e.,  . In other words we want to eliminate the possibility of having a repeated element in our sets.

A useful technique in proving statements is proof by contradiction. The underlying idea is to assume the opposite of what we are trying to prove and show that this leads to an impossible situation. This implies that our assumption is wrong, well our assumption was the opposite of what we are trying to prove, so this means what we want to prove is true. In other words we are proving something is not not true.

Example: Show that in any simple graph on two or more vertices there *must* be two vertices which have the same degree.

Solution: Suppose that it is not true, i.e., suppose that there is a graph where all vertices have different degrees. Also for convenience let us suppose that we have n vertices. Then the possible degrees in the graph are $0, 1, \ldots, n-1$. Since there are n vertices, n possible degrees and all of them are distinct this implies that we must have

- a vertex of degree 0, i.e., a vertex not connected to any other vertex; and
- a vertex of degree $n-1$, i.e., a vertex connected to every other vertex.

But clearly we cannot have both of these vertices in the same graph. Therefore our assumption is false, and in particular there must be some two vertices with the same degree.

In the last lecture we looked at telling if two graphs were isomorphic. We saw that the way to show that two graphs were isomorphic was to produce a bijective $\phi : V(G) \to V(H)$ that preserves adjacency. But now suppose that we want to show that two graphs are *not* isomorphic. To do this we need to find some structure that one of the graphs have that the other doesn't. Some possibilities to look for.

- If the graphs are isomorphic they must have the same number of vertices and the same number of edges.
- The two graphs must have the same degree sequences (i.e., the list of degrees).
- The complement of the graphs must be isomorphic.
- Both graphs must either be connected of not connected.

- If the graphs are isomorphic then if G has a subgraph K H must also have a subgraph isomorphic to K.

These are only a few of the possibilities of things to look for. The list is nearly endless of things to look for. The nice thing is that to show two graphs are not isomorphic we only need to find a single property they don't agree on, so we usually don't have to go through the whole list of possibilities to show graphs are not isomorphic.

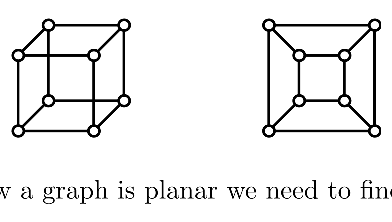
We have introduced a few terms on this list that we need to go over. The *complement* of a graph G, denoted $\overline{G}$ is the graph with the same set of vertices of G and all edges not in G. So for example the complement of $K_{m,n}$ would consist of a K_m and K_n with no edges between the two complete graphs. As another example, consider the complement of the Petersen graph, since K_{10} has degree 9 and the Petersen graph has degree 3 then the complement will be regular of degree 6 and so in particular will have 30 edges (we won't draw it). On a side note, the complement of the Petersen graph is the complement of the line graph of K_5 (but that is another story). If graphs have many edges sometimes it is easier to work with the complements which can have fewer edges.

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (\hat{V}, \hat{E})$ where $\hat{V} \subseteq V$ and $\hat{E} \subseteq E$. So for example a path on n vertices is a subgraph of a a cycle on n vertices which in turn is a subgraph of K_n (in fact every graph on n vertices is a subgraph of K_n).

A graph is *connected* if between any two vertices we can get from one vertex to another vertex by going along a series of edges. That is for two vertices u, v there is a sequence of vertices $v_0, v_1, \ldots, v_k$ so that $u = v_0 \sim v_1$, $v_1 \sim v_2$, $\ldots$, $v_{k-1} \sim v_k = v$.

Another type of graph is a *planar* graph. This is a graph which can be drawn in the plane in such a way so that no two edges intersect. Of course every graph can be drawn in the plane, the requirement that edges do not intersect though adds some extra constraint.

It is important to note in this definition that a graph is planar if there is some way to draw it in the plane without edges intersecting, it does not mean that every way to draw it in the plane avoids edges intersecting. So for example in the last lecture we saw the graph Q_3 and drew it as shown below on the left. In this drawing we have edges crossing each other twice so that this is not a way to embed it in the plane without edges crossing, but we can draw it in a way without edges crossing which is shown below on the right.



To show a graph is planar we need to find a way to draw it in the plane without any edges crossing. How do we show a graph is not planar. For instance try drawing $K_{3,3}$ or K_5 in the plane, it turns out to be difficult. But is it difficult because it is impossible or is it difficult because we are not seeing some clever way to draw it. (It turns out that in this case it is difficult because it is impossible, as we will soon prove.)

The main tool for dealing with planar graphs is Euler's formula. Before we state it we first need to give one more definition. So for a drawing of a planar graph we have vertices and edges, but we also have subdivided the plane into pieces which we call faces. If we were drawing the planar graph on a piece of paper then cutting along the edges we would cut our paper into some small pieces, each piece corresponds to a face. Let V denote the set of vertices, E the set of edges and F the set of faces in the drawing of a planar graph.

Euler's formula:
For a connected planar graph
$$|V| - |E| + |F| = 2.$$

As an example in the drawing of Q_3 above we have $|V| = 8$, $|E| = 12$ and $|F| = 6$ (remember to count the "outer" face). So $8 - 12 + 6 = 2$, as the formula predicted. In the next lecture we will prove Euler's formula and use it to show that the graphs K_5 and $K_{3,3}$ are not planar as well as give some other applications.

Lecture 20 – May 15

We now prove Euler's formula. The key fact is the following. Every connected planar graph can be drawn by first starting with a single isolated vertex and then doing a series of two types of operations, namely:

1. add a new vertex and an edge connecting the new vertex to the existing graph which does not cross any existing edge; and
2. add an edge between two existing vertices which does not cross any existing edge.

One way to see this is that we can run this backwards, i.e., starting with our drawing of a planar connected graph we either delete a vertex and its incident edge if it has degree one, if no such vertex exists we remove an edge from the graph which will not disconnect the graph.

(There is a subtlety to worry about, how do we know that if all vertices have degree two or more that we can remove an edge without disconnecting the graph? To see this note that if there were a cycle in the graph (i.e., a subgraph which is a cycle, or if you prefer a sequence of edges that starts and ends at the same point) then removing an edge could not disconnect the graph since we could simply use the rest of the cycle to replace the now missing edge. Further we can grow a cycle by the following technique, we start with any vertex and move to an adjacent vertex, we then move to a new vertex other than the one we came from and continue doing this until we get to a vertex that we have previously visited. So suppose that we now have a sequence of vertices

$$v_1 \sim v_2 \sim v_3 \sim \cdots \sim v_j$$

and that $v_j \sim v_i$ for some $i < j$. Then the desired cycle is

$$v_i \sim v_{i+1} \sim \cdots \sim v_j \sim v_i.$$

The key to this is since each vertex has degree two or more whenever we went into a vertex we could also exit along a different edge.)

We now proceed by induction. Our base graph consists of a single vertex. In that case we have $|V| = 1$, $|E| = 0$ and $|F| = 1$ (i.e., the outer face) and so

$$|V| - |E| + |F| = 1 - 0 + 1 = 2.$$

Now suppose that we have shown that we have $|V| - |E| + |F| = 2$ for our connected planar graph G. Suppose that we do the first type of operation as given above. Then when we are adding a new vertex and a new edge, but we do not change the number of faces (i.e., this edge does not divide a face), and so $|V'| = |V| + 1$, $|E'| = |E| + 1$ and $|F'| = |F|$ so

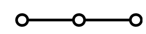
$$\begin{aligned} |V'| - |E'| + |F'| &= (|V| + 1) - (|E| + 1) + |F| \\ &= |V| - |E| + |F| = 2. \end{aligned}$$

Suppose that we do the second type of operation as given above. Then we do not add a vertex but we do add an edge, further we must split a face into two faces, and so $|V'| = |V|$, $|E'| = |E| + 1$ and $|F'| = |F| + 1$ so

$$\begin{aligned} |V'| - |E'| + |F'| &= |V| - (|E| + 1) + (|F| + 1) \\ &= |V| - |E| + |F| = 2. \end{aligned}$$

Concluding the proof.

Euler's formula is the main tool for dealing with planar graphs. In this lecture we will show how to use it to prove that some graphs are *not* planar, we will also see some additional applications in the next lecture. So let us first try to prove that the graph K_5 is not planar. Before we do this we first need to introduce the idea of counting the (bounding) edges in a face. Every face can be though of as being contained inside of some cycle and so we can count how many edges are in the cycle by starting at a vertex and walking clockwise around the face and counting how many edges we encounter before we return to our starting position, this will be the number of edges in the face. There is a small bit of subtlety to be aware of, namely consider the following graph, a planar graph on three vertices.



This graph has two edges and a single face (so as predicted by Euler's formula $3-2+1 = 2$). How many edges are in the face? Our natural instinct is to say two, but the correct answer is four. That is because as we walk around we will encounter each edge twice and we count an edge each time we encounter it. Put another way if we had cut along the edges we could not distinguish between this graph and what would have happened if we started with a four cycle.

The reason that we are using this definition is because we want to be able to count edges by using information about faces. Namely since each edge will be used in two faces (or twice in a single face) if we add up the number of edges in each face we will have double counted all the edges in the graph, i.e.,

$$2|E| = \sum_{f \in F} \left| \begin{matrix} \text{edges in} \\ \text{face } f \end{matrix} \right|.$$

We now make an observation, for every simple connected planar graph on at least three vertices, each face has at *least* three edges in it. (The reason that we limit ourselves to simple graphs is that a loop would be a face with a single edge while a pair of vertices connected by two edges would give a face bounded by two edges.) Plugging this into the above formula we have

$$2|E| = \sum_{f \in F} \left| \begin{matrix} \text{edges in} \\ \text{face } f \end{matrix} \right| \geq \sum_{f \in F} 3 = 3|F|,$$

so that $|F| \leq (2/3)|E|$. Putting this into Euler's formula we have

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E|$$

or rearranging we have

$$|E| \leq 3|V| - 6.$$

We are now ready to prove the following.

K_5 is not planar.

Suppose that K_5 were planar. Since it a simple connected graph on five vertices this would then imply by the above inequality that $|E| \leq 3|V| - 6$ but for K_5 we have $|E| = 10$ and $3|V| - 6 = 9$ so this is impossible. So K_5 is not planar.

Another graph that is difficult to draw in the plane is $K_{3,3}$, if we apply the above inequality we see that $|E| = 9$ and $3|V| - 6 = 12$. From this we can't conclude anything (just because a graph satisfies the above inequality does not automatically imply planar). We will have to work a little harder for $K_{3,3}$. One thing that we might be able to use is that the graph $K_{3,3}$ is bipartite, and bipartite graphs have nice structure. In particular we have the following.

A graph is bipartite if and only if every cycle (allowing for repetition of edges) in the graph has even length.

The assumption that we allow for edges to be repeated is not important and can be removed from the following arguments with a little care. Since it is an if and only if statement we need to prove both directions.

First let us suppose that our graph is bipartite. So we have that the vertices are in sets U and W with all edges going between U and W. Suppose that we start our cycle at a vertex in U, then after the first step we must be in a vertex in W, then after the next step we are back in U and so on. So our cycle keeps going back and forth between the two sets. After an odd number of steps we will be in W and after an even number of steps we will be in U. Since a cycle must begin and end at the same vertex our cycle must have even length.

Now let us suppose that every cycle in our graph has even length. Without loss of generality we may assume that our graph is connected (i.e., we can work on each component, a maximal connected subgraph, show that each is bipartite and so the whole graph is bipartite). We fix a vertex $\overline{v}$ and partition the vertices into V_{odd} and V_{even} by putting a vertex v into V_{odd} if there is a path of *odd* length between $\overline{v}$ and v and putting a vertex v into V_{even} if there is a path of *even* length between $\overline{v}$ and v. We claim that this is well defined, i.e., no vertex could be in both sets since if there were a path of even length and a path of odd length between $\overline{v}$ and v we could form a cycle by starting along the path of even length from $\overline{v}$ to v and then follow the path of odd length backwards from v to $\overline{v}$ (i.e., we are concatenating the paths). This new cycle would have an odd number of edges but by our assumption this is impossible. Similarly there cannot be an edge connecting two vertices v, w in V_{odd} (similarly for V_{even} since we could then form an odd cycle by going from $\overline{v}$ to v, from v to w and from w to $\overline{v}$. So this shows that we can split the vertices into two sets and all edges in the graph go between these sets, by definition this is a bipartite graph.

Since the edges bounding a face correspond to a cycle we must now be able to conclude that in a simple connected planar bipartite graph on at least three vertices each face has at least four edges, i.e.,

$$2|E| = \sum_{f\in F} \left| \begin{matrix}\text{edges in}\\ \text{face } f\end{matrix} \right| \geq \sum_{f\in F} 4 = 4|F|,$$

so that $|F| \leq (1/2)|E|$. Putting this into Euler's formula we have

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{1}{2}|E|$$

or rearranging we have

$$|E| \leq 2|V| - 4.$$

We are now ready to prove the following.

$K_{3,3}$ is not planar.

Suppose that $K_{3,3}$ were planar. Since it a simple connected bipartite graph on six vertices this would then imply by the above inequality that $|E| \leq 2|V| - 4$ but for $K_{3,3}$ we have $|E| = 9$ and $2|V| - 4 = 8$ so this is impossible. So $K_{3,3}$ is not planar.

We have now shown that $K_{3,3}$ and K_5 are not planar. In some sense these are *the* problem graphs for planarity. We will make this more precise in the next lecture.