

## Lecture 11 – April 24

It frequently happens that we want to prove a statement is true for  $n = 1, 2, \dots$ . For instance on the first day we saw that

$$1 + 2 + \dots = \frac{n(n+1)}{2}, \quad \text{for } n \geq 1.$$

We gave a proof for this using a method of counting a set of dots in two different ways. However we could prove it in another way. First let us prove it for  $n = 1$ . Since  $1 = 1(1+1)/2$ , the case  $n = 1$  is true. Now let us assume that the statement is true for  $n$  (i.e., we can assume it is true  $1 + 2 + \dots + n = n(n+1)/2$ ) and consider the case  $n + 1$ . We have

$$\begin{aligned} \underbrace{1 + 2 + \dots + n}_{\text{case for } n} + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1) \left( \frac{n}{2} + 1 \right) = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This shows that the case  $n + 1$  is also true. Between these two statements this shows it is true for  $n = 1, 2, 3, \dots$ . (I.e., we have shown it is true for  $n = 1$ , then by the second part since it is true for  $n = 1$  it is true for  $n = 2$ , then again by the second part since it is true for  $n = 2$  it is true for  $n = 3, \dots$ )

This is an example of *mathematical induction* which is a useful method of proving statements for  $n = 1, 2, \dots$  (though we don't have to start at 1 we can start at any value). In general we need to prove two parts:

1. The statement holds in the first case.
2. Assuming the statement holds for the cases  $k \leq n$ , show that the statement also hold for  $n + 1$ .

(This is what is known as *strong induction*. Usually it will suffice for us to only assume the previous case, i.e., assume the case  $k = n$  is true and then show  $n + 1$  also holds (which is known as *weak induction*.)

The idea behind proof by induction is analogous to the idea of climbing a ladder. We first need to get on the ladder (prove the first case), and then we need to be able to move from one rung of the ladder to the next (show that if one case is true then the next case is true. It is important that you remember to prove the first case!

The *most* important thing about an induction proof is to see how to relate the previous case(s) to the current case. Often once that is understood the rest of the proof is very straightforward.

*Example:* Prove for  $n \geq 1$ ,

$$-1^2 + 2^2 - 3^2 + \dots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}.$$

*Solution:* We first prove the case  $n = 1$ . Since

$$-1^2 = -1 = (-1)^1 \frac{1(1+1)}{2},$$

the first case holds. Now we assume the case corresponding to  $n$  is true, and now we want to prove the case corresponding to  $n + 1$  is true. That is, we assume

$$-1^2 + 2^2 - 3^2 + \dots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}.$$

and we want to show

$$\begin{aligned} -1^2 + 2^2 - 3^2 + \dots + (-1)^n n^2 + (-1)^{n+1} (n+1)^2 \\ = (-1)^{n+1} \frac{(n+1)(n+2)}{2}. \end{aligned}$$

So let us begin, we have

$$\begin{aligned} \underbrace{-1^2 + 2^2 - 3^2 + \dots + (-1)^n n^2}_{\text{case corresponding to } n} + (-1)^{n+1} (n+1)^2 \\ = (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1} (n+1)^2 \\ = (-1)^{n+1} (n+1) \left( -\frac{n}{2} + (n+1) \right) \\ = (-1)^{n+1} \frac{(n+1)(n+2)}{2}. \quad \square \end{aligned}$$

*Example:* Prove for  $n \geq 1$ ,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

*Solution:* We first prove the case  $n = 1$ . Since

$$\frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1},$$

the first case holds. Now we assume the case corresponding to  $n$  is true, and now we want to prove the case corresponding to  $n + 1$  is true. That is, we assume

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

and we want to show

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}.$$

So let us begin, we have

$$\begin{aligned} \underbrace{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)}}_{\text{case corresponding to } n} + \frac{1}{(n+1) \cdot (n+2)} \\ = \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n(n+2) + 1}{(n+1) \cdot (n+2)} \\ = \frac{(n+1)^2}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}. \quad \square \end{aligned}$$

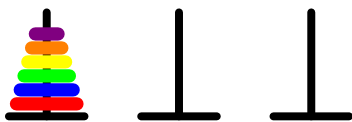
We now turn to recurrence relations. A recurrence relation (sometimes also referred to as a recursion relation) is a sequence of numbers where  $a(n)$ , sometimes written  $a_n$ , (the  $n$ th number) can be expressed using previous  $a(k)$ , for  $k < n$ . Examples of recurrence relations include the following:

- $a_n = 3a_{n-1} + 2a_{n-2} - a_{n-3}$ . This is an example of a constant coefficient, homogeneous linear recurrence. We will examine these in more detail in a later lecture.
- $a_n = 2a_{n-1} + n^2$ . This is an example of a non-homogeneous recursion. We will also examine these (in some special cases) in a later lecture.
- $a_{n+1} = a_0a_n + a_1a_{n-1} + \dots + a_na_0$ . This is related to the idea of *convolution* in analysis. It shows up frequently in combinatorial problems, in particular the Catalan numbers can be found using this method.
- $a_{n,k} = a_{n-1,k-1} + a_{n-1,k}$ . This is an example of a multivariable recurrence. That is we have that the number that we are interested in finding has two parameters instead of one. We have seen this particular recurrence before, namely,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We are often interested in *solving a recurrence* which is to find an expression for  $a(n)$  that is independent of the previous  $a(k)$ . In other words we want to find some function  $f(n)$  so that  $a(n) = f(n)$ . In order for us to do this we need one more thing, that is *initial conditions*. These initial conditions are used to “prime the pump” and get the recursion started.

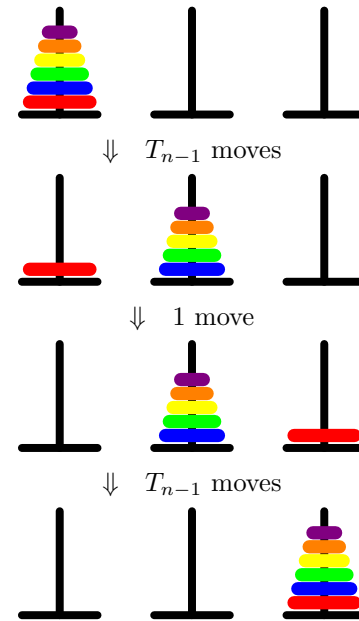
*Example:* The “Tower of Hanoi” puzzle relates to a problem of moving a stack of  $n$  differently sized discs on one pole to another pole, see the following picture.



There are two rules: (1) only one disc at a time can be moved; (2) at no time can a larger disc be placed over a smaller disc. Find a recursion for  $T_n$  the minimal number of moves needed to move  $n$  discs.

*Solution:* Experimenting we note that  $T_1 = 1$  (i.e., move the disc and we are done),  $T_2 = 3$  (i.e., move the small disc to the center pole, move the large disc to the end pole and then move the small disc back). We can keep this going, but now let us step back. As we are moving discs we must at some point move the

bottom disc from the left pole to the right pole. We can use this to break the operation of moving all the discs into several steps as outlined below.



So combining we have

$$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1.$$

We have now found a recursion for  $T_n$ . Using this (along with the first two cases we did by hand) we have the following values for  $T_n$ .

$n$	1	2	3	4	5	6	7	8	9	10
$T_n$	1	3	7	15	31	63	127	255	511	1023

Staring at the  $T_n$  they look a little familiar. In particular they are very close to the numbers 2, 4, 8, 16, 32, ... which are the powers of 2. So we can guess that  $T_n = 2^n - 1$ .

This right now is a guess. To show that the guess is correct we need to do the following two things:

- Show that the initial condition(s) are satisfied.
- Show that the recurrence relationships is satisfied.

(If this reminds you of induction, you are right! Showing the initial condition(s) is satisfied establishes the base case and showing that the recurrence relationship is satisfied shows that if it is true to a certain case, then the next case is also true.)

So we now check. Our initial condition is that  $T_1 = 1$ , and since  $2^1 - 1 = 1$  our initial condition is satisfied. Now we check that  $T_n = 2^n - 1$  satisfies the recurrence  $T_n = 2T_{n-1} + 1$ . We have

$$2T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 2 + 1 = 2^n - 1 = T_n$$

showing that the recurrence relationship is satisfied. This establishes that  $T_n = 2^n - 1$ . This is good news

since legend has it that there is a group of monks working on a copy of the Tower of Hanoi puzzle with 64 golden discs, when they are done the world will end. We now know that it will take them

$$2^{64} - 1 = 18,446,744,073,709,551,615 \text{ moves,}$$

so the world is in no danger of ending soon!

A very famous recursion is the one related to *Fibonacci numbers*. This recursion traces back to a problem about rabbit population in *Liber Abaci* one of the first mathematical textbooks about arithmetic written in 1202 by Leonardo de Pisa (also known as Fibonacci). The numbers are defined by  $F_0 = 0$ ,  $F_1 = 1$  and  $F_{n+2} = F_{n+1} + F_n$ , i.e., to find the next number add the two most recent numbers. The first few Fibonacci numbers are listed below. (Note: the book defines the Fibonacci numbers by letting  $F_0 = F_1 = 1$ , this is the same set of numbers but the index is shifted by 1; the definition we have given here is the more commonly accepted version.)

$n$	0	1	2	3	4	5	6	7	8	9	10
$F_n$	0	1	1	2	3	5	8	13	21	34	55

We will study these numbers in some detail and find some “simple” expressions for calculating the  $n$ th Fibonacci number in later lectures. The Fibonacci numbers are one of the most studied numbers in mathematics (they even have their own journal dedicated to studying them). This is in part because they arise in many different interpretations.

*Example:* Count the number of binary sequences with no consecutive 0s.

*Solution:* Let  $a(n)$  be the number of binary sequences of length  $n$  with no consecutive 0s. Calculating the first few cases we see the following.

$n$	admissible sequences	$a(n)$
0	$\emptyset$	1
1	0, 1	2
2	01, 10, 11	3
3	010, 011, 101, 110, 111	5
4	0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111	8

The sequence  $a(n)$  looks like Fibonacci numbers, in particular it looks like  $a(n) = F_{n+1}$ . But we still have to prove that. Let us turn to finding a recursion for the  $a(n)$ .

We can group the sequences without consecutive 0s of length  $n$  into two groups. The first group will be those that have a 1 at the end and the second group will have a 0 at the end. In the first group the first  $n - 1$  terms is any sequence of length  $n - 1$  without

consecutive 0s. In the second group, if the last term is 0 then we must have that the second to last term be 1 so the first  $n - 2$  terms is any sequence of length  $n - 2$  without consecutive 0s. In particular we have the following two groups

$$\underbrace{* * * \cdots * *}_{\text{length } n-1 \text{ such } a(n-1)} 1 \quad \text{and} \quad \underbrace{* * * \cdots *}_\text{length } n-2 \text{ such } a(n-2) 1 0$$

So by the rule of addition it follows that

$$a(n) = a(n - 1) + a(n - 2).$$

This is the same recurrence as the Fibonacci numbers, so we have that the number of such sequences are counted by the Fibonacci numbers!

*Example:* A *composition* is an ordered partition, so for example the compositions of 4 are:

$$1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2, 1+3, 3+1, 4$$

Count the number of compositions of  $n$  with all parts odd.

*Solution:* Let  $b(n)$  be the number of compositions of  $n$  with all parts odd. Calculating the first few cases we see the following.

$n$	admissible sequences	$b(n)$
1	1	1
2	1 + 1	1
3	1 + 1 + 1, 3	2
4	1 + 1 + 1 + 1, 1 + 3, 3 + 1	3
5	1 + 1 + 1 + 1 + 1, 1 + 1 + 3, 1 + 3 + 1, 3 + 1 + 1, 5	5

Again the sequence looks like Fibonacci numbers and in particular it looks like  $b(n) = F_n$ . Again we still have to prove it. (After all it is possible that something looks true for the first few billion cases but eventually is false.)

We can group compositions according to the size of the last part. There are two cases, if  $n$  is odd the last part can be any of  $n, n - 2, n - 4, \dots, 1$ . On the other hand if  $n$  is even the last part can be any of  $n - 1, n - 3, \dots, 1$ . In particular it is not difficult to see

$$b(n) = \begin{cases} 1 + b(n - 1) + b(n - 3) + \cdots & \text{if } n \text{ odd;} \\ b(n - 1) + b(n - 3) + \cdots & \text{if } n \text{ even.} \end{cases}$$

(The  $b(n - 1)$  term is when the last part is 1, the  $b(n - 3)$  is when the last part is 3, and so on. The 1 term for the case  $n$  odd corresponds to the composition  $n$ .)

This doesn't look like the recurrence for the Fibonacci numbers at all! But perhaps we can massage this recurrence and get the Fibonacci recurrence to

pop out. In particular we note that for the case  $n$  odd we have

$$\begin{aligned} b(n) &= 1 + b(n-1) + b(n-3) + b(n-5) + \dots \\ &= b(n-1) + \underbrace{(1 + b(n-3) + b(n-5) + \dots)}_{=b(n-2)} \\ &= b(n-1) + b(n-2). \end{aligned}$$

For the case  $n$  even we have

$$\begin{aligned} b(n) &= b(n-1) + b(n-3) + b(n-5) + \dots \\ &= b(n-1) + \underbrace{(b(n-3) + b(n-5) + \dots)}_{=b(n-2)} \\ &= b(n-1) + b(n-2). \end{aligned}$$

In particular, we see that  $b(n) = b(n-1) + b(n-2)$ , so the sequence  $b(n)$  does satisfy the same recurrence property as the Fibonacci numbers, and so we have that the  $b(n)$  are the Fibonacci numbers.

## Lecture 12 – April 27

*Example:* Find the number of compositions of  $n$  with no part equal to 1.

*Solution:* Let  $d(n)$  denote the number of such compositions for  $n$ . Then for the first few cases we have:

$n$	admissible sequences	$d(n)$
1		0
2	2	1
3	3	1
4	2 + 2, 4	2
5	2 + 3, 3 + 2, 5	3
6	2 + 2 + 2, 2 + 4, 3 + 3, 4 + 2, 6	5

Looking at the numbers 0, 1, 1, 2, 3, 5 these look familiar they look like Fibonacci numbers. (Again? Haven't we seen them enough? No, you can never see enough Fibonacci numbers!)

So now let us turn to making a recursion for  $d(n)$ . We can group compositions of  $n$  according to the size of the last part. Namely, the size of the last part can be  $2, 3, \dots, n-2, n$  (we cannot have the last part be size 1 since we are not allowed parts of 1, for the same reason we cannot have a part of size  $n-1$  since the other part would be 1). If the last part is  $k$  then we have that the rest of the composition would be some composition of  $n-k$  with no part of size 1. In particular it follows that

$$d(n) = d(n-2) + \underbrace{d(n-3) + d(n-4) + \dots + d(2) + 1}_{=d(n-1)}$$

in particular if we group all but the first term, what we have is  $d(n-1)$  and so

$$d(n) = d(n-2) + d(n-1).$$

The same recursion for the Fibonacci numbers. Since we start with Fibonacci numbers and the same recurrence is satisfied it follows that  $d(n) = F_{n-1}$ , i.e., the number of such compositions is counted by the Fibonacci numbers (with the index shifted by 1).

We now want to look at some important numbers that arise in combinatorics. We look at them now since they can be described using multivariable recurrences.

The *Stirling numbers of the second kind*, denoted  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ , counts the number of ways to partition the set  $\{1, 2, \dots, n\}$  into  $k$  disjoint nonempty subsets (i.e., each element is in one and only one subset). For example, we have the following.

$n, k$	possible partitions	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$
$n=4$ $k=1$	$\{1, 2, 3, 4\}$	1
$n=4$ $k=2$	$\{1, 2, 3\} \cup \{4\}, \{1, 2, 4\} \cup \{3\}$ $\{1, 3, 4\} \cup \{2\}, \{1\} \cup \{2, 3, 4\}$ $\{1, 2\} \cup \{3, 4\}, \{1, 3\} \cup \{2, 4\}$ $\{1, 4\} \cup \{2, 3\}$	7
$n=4$ $k=3$	$\{1\} \cup \{2\} \cup \{3, 4\}, \{1\} \cup \{2, 3\} \cup \{4\}$ $\{1\} \cup \{2, 4\} \cup \{3\}, \{1, 2\} \cup \{3\} \cup \{4\}$ $\{1, 3\} \cup \{2\} \cup \{4\}, \{1, 4\} \cup \{2\} \cup \{3\}$	6
$n=4$ $k=4$	$\{1\} \cup \{2\} \cup \{3\} \cup \{4\}$	1

The first few values of  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  are shown in the following table.

	$\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} n \\ 4 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} n \\ 5 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} n \\ 6 \end{smallmatrix} \right\}$
$n = 1$	1					
$n = 2$	1	1				
$n = 3$	1	3	1			
$n = 4$	1	7	6	1		
$n = 5$	1	15	25	10	1	
$n = 6$	1	31	90	65	15	1

Looking at this table we see a few patterns pop out.

- $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = 1$ . This is obvious since the only way to break a set into one subset is to take the whole set.
- $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1$ . This is also obvious since the only way to break a set with  $n$  elements into  $n$  subsets is to have each element of the set in a subset by itself.
- $\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\} = 2^{n-1} - 1$ . To see this note that if we break our set into two sets then it has the form of

$A \cup \bar{A}$  (i.e.,  $A$  and its complement). We may assume that 1 is in  $A$ . For each remaining element it is either in  $A$  or not, the only condition is that they cannot all be in  $A$  (else  $\bar{A} = \emptyset$ ), so there are  $2^{n-1} - 1$  ways to fill in the remainder of  $A$ .

- $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$ . This can be seen by noting that there is one subset with two elements with two elements and the remaining subsets are singletons (i.e., subsets of size one). So the number of such partitions is the same as the number of ways to pick the elements in the subset with two elements which can be done in  $\binom{n}{2}$  ways.

On a side note the row sums of this table are 1, 2, 5, 15, 52, ... these are also important numbers known as the Bell numbers.

We now turn to the recursion for the Stirling numbers of the second kind.

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}.$$

(Note that this is similar to the recursion for binomial coefficients. The only difference being the additional factor of  $k$ .)

To verify the recurrence we break the partitions of  $\{1, 2, \dots, n\}$  into two groups. In the first group are partitions with the set  $\{n\}$  and in the second group are partitions without the set  $\{n\}$ . In the first group we have  $n$  in a set by itself and we need to partition the remaining  $n-1$  elements into  $k-1$  sets which can be done in  $\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$  ways. In the second group we first form  $k$  sets using the first  $n-1$  elements, which can be done in  $\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$  ways, we now need to add  $n$  into one of the sets, since there are  $n$  sets and we can put them into any one of them the number in this group is  $k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ .

The Stirling numbers of the second kind arise in many applications (even more than the Stirling numbers of the first kind). As an example consider the following.

*Example:* Show that the number of ways to place  $n-k$  non-attacking rooks below the diagonal of an  $n \times n$  is  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ .

*Solution:* Since we know that  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  counts the number of partitions of  $\{1, 2, \dots, n\}$  one method is to give a one-to-one correspondence between the rook placements and the set partitions. We will describe the positions of the chessboard as coordinates  $(i, j)$  with  $i < j$ . So suppose that

$$\{1, 2, \dots, n\} = A_1 \cup A_2 \cup \dots \cup A_k$$

is a partitioning, and suppose that

$$A_i = \{a_1, a_2, \dots, a_{m(i)}\} \text{ with } a_1 < a_2 < \dots < a_{m(i)}.$$

Then place rooks at the coordinates  $(a_1, a_2), (a_2, a_3), \dots, (a_{m(i)-1}, a_{m(i)})$ . For each set we will place  $|A_i| - 1$  rooks so altogether we will place

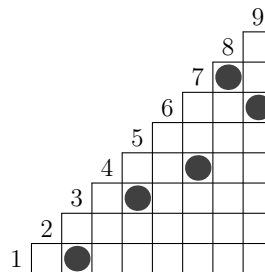
$$\sum_{i=1}^k (|A_i| - 1) = n - k \text{ rooks.}$$

Further note, that by this convention at most one rook will go in each row and in each column so that the rooks are non-attacking. Conversely, given a non-attacking rook placement we can reconstruct the partition. Namely, if there is a rook at position  $(p, q)$  then put  $p$  and  $q$  into the same subset of the partition. It is easy to see that each placement of rooks corresponds to a partition and each partition corresponds to a placement of rooks, giving us the desired correspondence.

The best way to see this is to work through a few examples. We will illustrate one here and encourage the reader to make up their own. So suppose that our partition is

$$\{1, 3, 5\} \cup \{2\} \cup \{4, 7, 8\} \cup \{6, 9\}.$$

Then the corresponding rook placement is shown below.



We also have *Stirling numbers of the first kind*, denoted  $\left[ \begin{matrix} n \\ k \end{matrix} \right]$ . These count the number of permutations in the symmetric group  $\mathcal{S}_n$  that have  $k$  cycles. Equivalently, this is the number of ways to sit  $n$  people at  $k$  circular tables so that no table is empty.

If we let  $(abc)$  denote a table with persons  $a, b$  and  $c$  seated in clockwise order (note that  $(abc) = (bca) = (cab)$  but  $(abc) \neq (acb)$ ), then we have the following.

$n, k$	possible seatings	$\left[ \begin{matrix} n \\ k \end{matrix} \right]$
$n=4$ $k=1$	(1234), (1243), (1324) (1342), (1423), (1432)	6
$n=4$ $k=2$	(1)(234), (1)(243), (2)(134) (2)(143), (3)(124), (3)(142) (4)(123), (4)(132), (12)(34) (13)(24), (14)(23)	11
$n=4$ $k=3$	(1)(2)(34), (1)(3)(24), (1)(4)(23) (12)(3)(4), (13)(2)(4), (14)(2)(3)	6
$n=4$ $k=4$	(1)(2)(3)(4)	1

The first few values of  $\left[ \begin{matrix} n \\ k \end{matrix} \right]$  are shown in the following table.

	$\left[ \begin{matrix} n \\ 1 \end{matrix} \right]$	$\left[ \begin{matrix} n \\ 2 \end{matrix} \right]$	$\left[ \begin{matrix} n \\ 3 \end{matrix} \right]$	$\left[ \begin{matrix} n \\ 4 \end{matrix} \right]$	$\left[ \begin{matrix} n \\ 5 \end{matrix} \right]$	$\left[ \begin{matrix} n \\ 6 \end{matrix} \right]$
$n = 1$	1					
$n = 2$	1	1				
$n = 3$	2	3	1			
$n = 4$	6	11	6	1		
$n = 5$	24	50	35	10	1	
$n = 6$	120	274	225	85	15	1

Looking at this table we again see a few patterns pop out.

- $\left[ \begin{matrix} n \\ 1 \end{matrix} \right] = (n - 1)!$ . This is obvious since everyone is sitting at one table. Fix the first person, then there are  $n - 1$  possible people to seat in the next seat,  $n - 2$  in the following seat and so on.
- $\left[ \begin{matrix} n \\ n \end{matrix} \right] = 1$ . This is obvious since the only way that this can happen is if everyone is at a private table.
- $\left[ \begin{matrix} n \\ n - 1 \end{matrix} \right] = \binom{n}{2}$ . As before, one table will have a pair of people sitting and the rest will have one person each. We only need to pick the two people who will be sitting at the table together, which can be done in  $\binom{n}{2}$  ways.

Looking at the the row sums of this table we have 1, 2, 6, 24, 120, 720, . . . , which are the factorial numbers. This is easy to see once we note that there is a one-to-one correspondence between permutations and these seatings. (Essentially the idea here is that we are refining our count by grouping permutations according to the number of cycles.)

We now turn to the recursion for the Stirling numbers of the first kind.

$$\left[ \begin{matrix} n \\ k \end{matrix} \right] = \left[ \begin{matrix} n - 1 \\ k - 1 \end{matrix} \right] + (n - 1) \left[ \begin{matrix} n - 1 \\ k \end{matrix} \right].$$

Again we note the similarity to the recursion for the binomial coefficients as well as the Stirling numbers of the second kind. Of course a small change can lead to very different behavior!

To verify the recurrence we break the seating arrangements into two groups. In the first group we consider the case where  $n$  sits at a table by themselves and in the second group  $n$  sits at a table with other people. In the first group since  $n$  is at a table by themselves this leaves  $n - 1$  people to fill in the other  $k - 1$  tables which can be done in  $\left[ \begin{matrix} n - 1 \\ k - 1 \end{matrix} \right]$  ways. In the second group first we seat everyone besides  $n$  at  $k$  tables, this can be done in  $\left[ \begin{matrix} n - 1 \\ k \end{matrix} \right]$  ways, now we let  $n$  sit down, since  $n$  can sit to the left of any person there are  $n - 1$

places that they could sit, and so the total number of arrangements in this group is  $(n - 1) \left[ \begin{matrix} n - 1 \\ k \end{matrix} \right]$ .

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## Lecture 13 – April 29

Previously we saw that if we can guess the solution to a recurrence problem then we can prove that it is correct by verifying the guess satisfies the initial conditions and satisfies the recurrence relation. But this assumes we can guess the solution which is not easy. For instance what is a way to express the Fibonacci numbers without using the recursion definition? We now turn to systematically finding solutions to some recurrences.

In this lecture we limit ourselves to solving *homogeneous constant coefficient linear recursions of order  $k$* . This is a mouthful of adjectives, with this many assumptions we should be able to solve these recurrences! Recursions of this type have the following form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}, \quad (*)$$

where the  $c_1, c_2, \dots, c_k$  are constants, hence the term “constant coefficient”. That we only look back in the previous  $k$  terms to determine the next term explains the “order  $k$ ”, that we are using linear combinations of the previous  $k$  terms explains the “linear”. Finally “homogeneous” tells us that there is nothing else besides the linear combination of the previous  $k$  terms, an example of something which is not homogeneous would be

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n),$$

where  $f(n)$  is some nonzero function depending on  $n$ .

Our method, and even the language that we use, in solving these linear recurrences is similar to the approach in solving differential equations. So for example when we had a homogeneous constant coefficient linear differential equation of order  $k$ , we would try to find solutions of the form  $y = e^{rx}$  and determine which values of  $r$  are possible. Are approach will be the same, we will try to find solutions of the form  $a_n = r^n$  and determine which values of  $r$  are possible. Putting this into the equation (\*) we get

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

Dividing both sides of this equation by the smallest power of  $r$  (in this case  $r^{n-k}$ ) this becomes

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k,$$

or  $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0.$

Because the  $c_i$ 's are constants this last expression does not depend on  $n$ , but is a  $k$ th degree polynomial. So we can find the roots of this polynomial (at least in

theory!), and the roots are the possible values of  $r$ . So suppose that we have the roots  $r_1, r_2, \dots, r_k$  (for now we will assume that they are all distinct). We now have  $k$  different solutions and so we can combine them together in a linear combination (hence the important reason that we assumed linear). So that some solutions are of the form

$$a_n = D_1 r_1^n + D_2 r_2^n + \dots + D_k r_k^n,$$

where  $D_1, D_2, \dots, D_k$  are constants. Now let us back to the original problem. Equation (\*) tells us that we look at the previous  $k$  terms to determine the next term. So in order for the recursion to get started we need to have at least  $k$  initial terms (the initial conditions). In other words we have  $k$  degrees of freedom, and conveniently enough we have  $k$  constants available. So using the  $k$  initial conditions we can solve for the  $k$  constants  $D_1, D_2, \dots, D_k$ . In fact, every solution to this type of recurrence must be of this type!

We still need to determine what to do when there are repeated roots, we will address this in the next lecture.

*Example:* Solve the recurrence relationship  $a_0 = 4$ ,  $a_1 = 11$  and  $a_n = 2a_{n-1} + 3a_{n-2}$  for  $n \geq 2$ .

*Solution:* First we translate the recurrence relationship into a polynomial in  $r$ . Since it is a second order recurrence this will become a quadratic, namely

$$r^2 = 2r + 3 \quad \text{or} \quad 0 = r^2 - 2r - 3 = (r - 3)(r + 1).$$

So that the roots are  $r = -1, 3$ . So that the general solutions are

$$a_n = C(-1)^n + D3^n.$$

The initial conditions translate into

$$\begin{aligned} 4 &= a_0 &= C + D, \\ 11 &= a_1 &= -C + 3D. \end{aligned}$$

Adding these together we have  $15 = 4D$  or  $D = 15/4$ , substituting this into the first equation we have  $C = 4 - 15/4 = 1/4$ . So our solution to the recurrence is

$$a_n = \frac{1}{4}(-1)^n + \frac{15}{4}3^n.$$

*Example:* Recall that the Fibonacci numbers are defined by  $F_0 = 0$ ,  $F_1 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for  $n \geq 0$ . Find an expression for the  $n$ th Fibonacci number.

*Solution:* First we translate the recurrence relationship into a polynomial in  $r$ . This is again a second order recurrence so it becomes a quadratic, namely

$$r^2 = r + 1 \quad \text{or} \quad r^2 - r - 1 = 0.$$

Since it is not obvious how to factor we plug this into the quadratic formula

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

So the general solution to this recursion is

$$F_n = C \left( \frac{1 + \sqrt{5}}{2} \right)^n + D \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

We now plug the initial conditions to solve for  $C$  and  $D$ . We have

$$\begin{aligned} 0 &= F_0 &= C + D \\ 1 &= F_1 &= C \left( \frac{1 + \sqrt{5}}{2} \right) + D \left( \frac{1 - \sqrt{5}}{2} \right) \\ & &= \frac{1}{2}(C + D) + \frac{\sqrt{5}}{2}(C - D) = \frac{\sqrt{5}}{2}(C - D) \end{aligned}$$

The first equation show that  $C = -D$ , putting this into the second equation we can now solve for  $C$  and  $D$ , namely  $C = 1/\sqrt{5}$  and  $D = -1/\sqrt{5}$ . Substituting these in we have

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

This seems very unusual, the Fibonacci numbers are whole numbers and so don't have any  $\sqrt{5}$  terms in them but the above expression is rife with them. So we should check our solution. One way is to plug it into the recurrence and verify that this works, but that can take some time. A quick way to check is to plug in the next term and verify that we get the correct answer. For example by the recursion we have that  $F_2 = 1$ , but plugging  $n = 2$  in the above expression we have

$$\begin{aligned} &\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \\ &= \frac{1 + 2\sqrt{5} + 5}{4\sqrt{5}} - \frac{1 - 2\sqrt{5} + 5}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1. \end{aligned}$$

Notice the  $\sqrt{5}$  terms all cancel out so that we are left with whole numbers.

The number that shows up in  $(1 + \sqrt{5})/2$  is called the golden ratio, denoted by  $\phi$ . This number is one of the most celebrated constants in mathematics and dates back to the ancient Greeks who believed that the best looking rectangle would be one that is similar to a  $1 \times \phi$  rectangle. The idea being that in such a rectangle if we cut off a  $1 \times 1$  square the leftover piece is similar to what we started with. However, experimental studies indicate that when asked to choose from a large group of rectangles that people tend not to go with the ones proportional to  $1 \times \phi$ . So perhaps the Greeks were wrong about that.

Finally, let us make one more observation about the Fibonacci numbers. We have

$$\frac{1 + \sqrt{5}}{2} = 1.618033988\dots$$

$$\frac{1 - \sqrt{5}}{2} = -0.618033988\dots$$

In particular since  $|(1 - \sqrt{5})/2| < 1$  when we take high powers of this term we have that this becomes very small. From this we have that the second term in the above expression for  $F_n$  is smaller than  $1/2$  for all values of  $n \geq 0$  and so

$$F_n = \text{nearest integer to } \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n.$$

So that the rate of growth of the Fibonacci numbers is  $\phi$ .

*Example:* Solve the recurrence  $g_n = 2g_{n-1} - 2g_{n-2}$  with the initial conditions  $g_0 = 1$  and  $g_1 = 4$ .

*Solution:* We again translate this into a polynomial in  $r$ . This becomes the quadratic

$$r^2 = 2r - 2 \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

Putting this into the quadratic formula we find that the roots are

$$r = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i.$$

Now we have complex roots! So we might pause and think how do we change our approach. The answer is not at all, the exact same techniques work whether the roots are real or complex. So now the general solution is

$$g_n = C(1 + i)^n + D(1 - i)^n.$$

Putting in the initial conditions this translates into

$$\begin{aligned} 1 = g_0 &= C + D. \\ 4 = g_1 &= C(1 + i) + D(1 - i) \\ &= (C + D) + i(C - D) = 1 + i(C - D). \end{aligned}$$

This gives  $C + D = 1$  and  $C - D = -3i$ . Adding together and dividing by 2 we have  $C = (1 - 3i)/2$  while taking the difference and dividing by 2 we have  $D = (1 + 3i)/2$ . So our solution is

$$g_n = \left( \frac{1 - 3i}{2} \right) (1 + i)^n + \left( \frac{1 + 3i}{2} \right) (1 - i)^n.$$

It should be noted that  $C$  and  $D$  are complex conjugates. This must happen in order to have the expression for  $g_n$  to be real values (which must be true by the recurrence).

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## Lecture 14 – May 1

As mentioned in the last lecture, solving recurrences is very similar to solving differential equations. Many of the same techniques that worked for differential equations will work for recurrences. As an example when solving the differential equation  $y'' - 2y' + y = 0$  we would turn this into a polynomial  $r^2 - r + 1 = 0$  and get the roots  $r = 1, 1$ , we would then translate this into the general solution  $y = Ce^x + Dxe^x$ , the extra factor of  $x$  comes from the fact that we have a double root, i.e., we could not use  $Ce^x + De^x = (C + D)e^x = C'e^x$  as this does not have enough freedom for giving general solutions to the differential equation.

For recurrences we will have essentially the same thing occur. Suppose that we are working with the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

then for finding the general solution we would look at the roots of

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0.$$

Suppose that  $\rho$  is a root of multiplicity  $\ell$ , then the general solution has the form

$$a_n = D_1 \rho^n + D_2 n \rho^n + D_3 n^2 \rho^n + \dots + D_\ell \rho^{\ell-1} \rho^n + \dots$$

where the “ $+\dots$ ” corresponds to the contribution of any other roots. Notice that we have multiplicity  $\ell$  and that we have  $\ell$  constants, just the right amount (this should always happen!).

*Example:* Let  $a_0 = 11$ ,  $a_1 = 6$  and  $a_n = 4a_{n-1} - 4a_{n-2}$  for  $b \geq 2$ . Solve the recurrence for  $a_n$ .

*Solution:* We first translate the recurrence into the polynomial  $r^2 - 4r + 4 = 0$  which has a double root of 2. So the general solution will be of the form

$$a_n = C2^n + Dn2^n.$$

To solve for  $C$  and  $D$  we use our initial conditions. We have  $a_0 = C = 11$  and  $a_1 = 2C + 2D = 6$  from which we can deduce that  $D = -8$ . So the solution is

$$a_n = 11 \cdot 2^n - 8n2^n.$$

*Example:* Solve the recurrence

$$b(n) = 3b(n-1) - 3b(n-2) + b(n-3)$$

with initial conditions  $b(0)=2$ ,  $b(1)=-1$  and  $b(2)=3$ .

*Solution:* We first translate the recurrence into the polynomial

$$0 = r^3 - 3r^2 + 3r - 1 = (r - 1)^3$$

which has a triple root of 1. So the general solution will be of the form

$$b(n) = C1^n + Dn1^n + En^21^n = C + Dn + En^2.$$

To solve for the constants we use our initial conditions. We have

$$\begin{aligned} 2 &= a_0 = C, \\ -1 &= a_1 = C + D + E, \\ 3 &= a_2 = C + 2D + 4E. \end{aligned}$$

From the first equation we have  $C = 2$ . Putting this in we have  $D + E = -3$  and  $2D + 4E = 1$ . Solving these we get  $D = -13/2$  and  $E = 7/2$ . So our solution is

$$b(n) = 2 - \frac{13}{2}n + \frac{7}{2}n^2.$$

We can occasionally take a recurrence problem which is not a constant coefficient homogeneous linear recurrence of order  $k$  and rewrite it so that it is of the form, thus allowing us to use the techniques that we have discussed so far to solve.

*Example:* Let  $d_0 = 1$  and  $nd_n = 3d_{n-1}$ . Solve for  $d_n$ .

*Solution:* This does not have constant coefficients so our current techniques do not work. One method to solve this is to start listing the first few terms and look for a pattern (in this case it is not hard to find the pattern!). Instead let us do something that at first blush seems to make things worse, let us take the recurrence and multiply both sides by  $(n - 1)!$  giving

$$(n - 1)!nd_n = n!d_n = 3(n - 1)!d_{n-1}.$$

Now let us make a quick substitution, namely let us set  $c_n = n!d_n$ . Then the above recurrence becomes  $c_n = 3c_{n-1}$  which has the solution  $c_n = n!d_n = E3^n$ , which shows that the general solution is  $d_n = E3^n/n!$ . Our initial condition then shows that  $1 = d_0 = E$  so that our desired solution is  $d_n = 3^n/n!$ .

The “trick” in the last example was to find a way to substitute to rewrite the recurrence as one that we have already done. This allowed us to take a recurrence which did not have constant coefficients and treat it like one that did. This same technique can also take some recurrences which are not linear and reduce it to ones which are linear.

*Example:* Find the solution to the recurrence with  $a_0 = 1$ ,  $a_2 = 2$  and

$$a_n = 2\sqrt{(a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2})} \text{ for } n \geq 2.$$

*Solution:* We start by rewriting this recurrence, note that for the term inside the square root it is of the form  $(a + b)(a - b)$  which we can replace by  $a^2 - b^2$ , doing this and squaring both sides we have

$$a_n^2 = 4a_{n-1}^2 - 4a_{n-2}^2.$$

There is an obvious substitution to make, namely  $b_n = a_n^2$  which relates this problem to the recurrence

$$b_n = 4b_{n-1} - 4b_{n-2},$$

with initial conditions  $b_0 = a_0^2 = 1$  and  $b_1 = a_1^2 = 4$ . As we saw in a previous example this has the general solution

$$b_n = C2^n + Dn2^n.$$

It is easy to see that the initial conditions translate into  $C = 1$  and  $D = 1$ . So we have

$$a_n^2 = b_n = 2^n + n2^n = (n + 1)2^n.$$

Taking square roots (and seeing that we want to go with the “+” square root to match our initial conditions) we have

$$a_n = \sqrt{(n + 1)2^n}.$$

*Example:* Solve for  $g_n$  where  $g_1 = 1$ ,  $g_2 = 2$  and

$$g_n = \frac{(g_{n-1})^2}{(g_{n-2})^{3/4}} \text{ for } n \geq 2.$$

*Example:* This is highly nonlinear. The problem is that we have division and then we also have things being raised to powers. We would like to translate this into something similar to what we have already done. Thinking back over our repertoire we recall that logarithms turn division into subtraction and also allow us to bring powers down. So let us take the logarithm of both sides. We could use any base that we want but let us go with  $\log_2$  (log base 2). Then we have

$$\log_2 g_n = 2 \log_2 g_{n-1} - \frac{3}{4} \log_2 g_{n-2}.$$

If we let  $h_n = \log_2 g_n$  then our problem translates into

$$h_n = 2h_{n-1} - \frac{3}{4}h_{n-2}$$

with initial conditions  $h_0 = \log_2 g_0 = 0$  and  $h_1 = \log_2 g_1 = 1$  (our choice of base 2 was made precisely so that these initial conditions would be clean). To solve this recurrence we first translate this into the polynomial

$$0 = r^2 - 2r + \frac{3}{4} = \left(r - \frac{1}{2}\right)\left(r - \frac{3}{2}\right),$$

so that our roots are  $r = 1/2, 3/2$ . So the general solution for  $h_n$  is

$$h_n = C\left(\frac{1}{2}\right)^n + D\left(\frac{3}{2}\right)^n.$$

Our initial conditions give us  $0 = C + D$  and  $1 = (1/2)C + (3/2)D$  or  $2 = C + 3D$ . It is easy to solve and find  $C = -1$  and  $D = 1$ . So we now have

$$\log_2 g_n = h_n = -\left(\frac{1}{2}\right)^n + \left(\frac{3}{2}\right)^n = \frac{3^n - 1}{2^n}.$$

So now solving for  $g_n$  we have

$$g_n = 2^{(3^n - 1)/2^n}.$$

Finally, let us look at how to deal with a recurrence relation which is non-homogeneous, i.e., a recurrence relation of the form.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n).$$

In differential equations there are several techniques to deal with this situation. We will be interested in looking at the technique known as the “method of undetermined coefficients”, or as I like to call it the “method of good guessing”. This technique has definite limitations in that we need to assume that  $f(n)$  is of a very specific form, namely

$$f(n) = \sum (\text{polynomial}) \cdot \rho^n,$$

i.e., that  $f(n)$  looks like what is possible as a homogeneous solution. (This is the same principle when doing the method of undetermined coefficients in differential equations.)

The outline of how to solve a non-homogeneous equation is as follows:

1. Solve the homogeneous part (i.e., the recurrence without the  $f(n)$ ).
2. Solve the non-homogeneous part by setting up a solution for  $a_n$  with some coefficients to be determined by the recurrence.
3. Combine the above two part to get the general solution. Solve for the constants using the initial conditions.

Note that the order of operations is important. That is, we need to solve for the homogeneous part before we can do the non-homogeneous part and we need to solve both parts before we can use initial conditions.

The reason that we need to solve the homogenous part first is that it can influence how we solve the non-homogeneous part. So now let us look at step 2 a little more closely. So suppose that we have

$$f(n) = (j\text{th degree polynomial in } n)\rho^n.$$

We look at the homogeneous part and see if  $\rho$  is a root, i.e., part of the homogeneous solution. If it is not a root then we guess that the non-homogeneous solution will be of the form

$$a_n = (B_j n^j + B_{j-1} n^{j-1} + \dots + B_0) \rho^n,$$

where  $B_j, B_{j-1}, \dots, B_0$  are constants which will be determined by putting this into the recursion, grouping coefficients and then making sure each coefficient is zero. (See the examples below.)

If  $\rho$  is a root then part of the above guess actually is in the homogeneous part and cannot contribute to the non-homogeneous part. In this case we need to gently nudge our solution. To do this, suppose that  $\rho$  occurs as a root  $m$  times. Then we modify our guess for the non-homogeneous solution so that it is now of the form

$$a_n = (B_j n^{m+j} + B_{j-1} n^{m+j-1} + \dots + B_0 n^m) \rho^n.$$

That is we multiply by a power of  $n^m$  to push the terms outside of the homogeneous solution.

If  $f(n)$  has several parts added together we essentially do each part separately and combine them together.

We now illustrate the approach with some examples.

*Example:* Solve for  $q_n$  where  $q_0 = -1$ ,  $q_1 = 1$  and

$$q_n = q_{n-2} + 3^n \text{ for } n \geq 2.$$

*Solution:* The term “ $+3^n$ ” shows that this is a non-homogeneous recurrence, and further  $3^n$  is of the form that we can do something with. So we first solve the homogeneous part  $q_n = q_{n-2}$  which turns into the polynomial  $r^2 = 1$  so that the roots are  $r = \pm 1$ . So the homogeneous portion has the form

$$q_n = C1^n + D(-1)^n = C + D(-1)^n.$$

Now neither of the roots are 3 and so we now set up the form for the non-homogeneous part. Namely

$$q_n = E3^n,$$

we now need to determine  $E$ . To do this we substitute into the recursion. This gives

$$E3^n = E3^{n-2} + 3^n \text{ or } \left(E - \frac{1}{9}E - 1\right)3^n = 0.$$

The second part comes from moving everything to one side and collecting the coefficients. In order for this last statement to hold (i.e., in order for it to be a solution to the non-homogeneous part) we need to have that the coefficient is 0. This means that we need  $(8/9)E - 1 = 0$  so that we need to choose  $E = 9/8$ . So the solution for the non-homogeneous part is

$$q_n = \frac{9}{8}3^n.$$

Combining the two parts, the general solution to this recurrence is

$$q_n = C + D(-1)^n + \frac{9}{8}3^n.$$

We now take care of the initial conditions. We have

$$\begin{aligned} -1 = q_0 &= C + D + \frac{9}{8} \\ 1 = q_1 &= C - D + \frac{27}{8} \end{aligned}$$

Rearranging this gives

$$\begin{aligned} -\frac{17}{8} &= C + D, \\ -\frac{19}{8} &= C - D. \end{aligned}$$

Adding the two we get  $2C = -36/8 = -9/2$  so that  $C = -9/4$ , taking the difference we get  $2D = 2/8 = 1/4$  so that  $D = 1/8$ . So our desired solution is

$$q_n = -\frac{9}{4} + \frac{1}{8}(-1)^n + \frac{9}{8}3^n.$$

*Example:* Let  $a_0 = 11$ ,  $a_1 = 8$  and

$$a_n = 3a_{n-1} - 2a_{n-2} + (-1)^n + n2^n.$$

Solve for  $a_n$ .

*Solution:* Again we have a non-homogeneous equation, and it is of the form that we can do something with. So we first solve the homogeneous part

$$a_n = 3a_{n-1} - 2a_{n-2}.$$

This will translate into the polynomial  $r^2 - 3r + 2 = 0$  which factors as  $(r - 2)(r - 1) = 0$ . So that the roots are  $r = 1, 2$ . So the solution to the homogeneous part is

$$a_n = C1^n + D2^n = C + D2^n.$$

We now turn to the non-homogeneous portion. Since  $(-1)$  is not a root of the homogeneous solution then that part of the solution will be of the form  $E(-1)^n$ . Unfortunately  $2$  is a root of the homogeneous solution (with multiplicity one) and so we will need to modify our guess, so instead of  $(Fn + G)2^n$  we will use  $(Fn^2 + Gn)2^n$ . So our non-homogeneous solution has the form

$$a_n = E(-1)^n + (Fn^2 + Gn)2^n.$$

We now substitute this in and get

$$\begin{aligned} E(-1)^n + (Fn^2 + Gn)2^n &= \\ 3(E(-1)^{n-1} + (F(n-1)^2 + G(n-1))2^{n-1}) & \\ - 2(E(-1)^{n-2} + (F(n-2)^2 + G(n-2))2^{n-2}) & \\ + (-1)^n + n2^n. & \end{aligned}$$

The next step is to expand and collect coefficients. If we do this we get the following:

$$\begin{aligned} 0 &= (-E - 3E - 2E + 1)(-1)^n \\ &+ \left(\frac{3}{2}F - \frac{3}{2}G - 2F + G\right)2^n \\ &+ \left(-G - 3F + \frac{3}{2}G + 2F - \frac{1}{2}G + 1\right)n2^n \\ &+ \left(-F + \frac{3}{2}F - \frac{1}{2}F\right)n^22^n \end{aligned}$$

Each of these coefficients must be zero in order for this to be a solution. This leads us to the following system of equations (note that the last coefficient is automatically zero and so we will drop it):

$$\begin{aligned} 1 &= 6E \\ 0 &= F + G \\ 1 &= F \end{aligned}$$

This is an easy system to solve. Giving  $E = 1/6$ ,  $F = 1$  and  $G = -1$  so the solution to the non-homogeneous part is

$$a_n = \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

So the general solutions is

$$a_n = C + D2^n + \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

It remains to use the initial conditions to find the constants  $C$  and  $D$ . Plugging in the initial conditions we have

$$\begin{aligned} 11 = a_0 &= C + D + \frac{1}{6} \\ 8 = a_1 &= C + 2D - \frac{1}{6} \end{aligned}$$

Giving  $C + D = 65/6$  and  $C + 2D = 49/6$ . Subtracting the second from the first we have that  $D = -16/6 = -8/3$  and so  $C = 81/6 = 27/2$ . So our final solution is

$$a_n = \frac{27}{2} - \frac{8}{3}2^n + \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

(Using this formula we get  $a_0 = 11$ ,  $a_1 = 8$ ,  $a_2 = 11$ ,  $a_3 = 40$ ,  $a_4 = 163$  which is what the recurrence says we should get. Wohoo!)

## Lecture 15 – May 4

We can use generating functions to help solve recurrences. The idea is that we are given a recurrence for  $a_n$ , and we want to solve for  $a_n$ . This is done by letting

$$g(x) = \sum_{n \geq 0} a_n x^n,$$

we then translate the recurrence into a relationship for  $g(x)$  which lets us solve for  $g(x)$ . We finally take the function  $g(x)$  and expand it to find the coefficient for  $x^n$ .

Broken into steps we would do the following for a recurrence with initial conditions  $a_0, a_1, \dots, a_{k-1}$  and a recurrence  $a_n = f(n, a_{n-1}, a_{n-2}, \dots)$  for  $n \geq k$ .

1. Write  $g(x) = \sum_{n \geq 0} a_n x^n$ .
2. Break off the initial conditions and use the recurrence to replace the remaining terms, i.e., so we have

$$g(x) = a_0 + a_1 x + \dots + a_{k-1} x^{k-1} + \sum_{n \geq k} f(n, a_{n-1}, a_{n-2}, \dots) x^n.$$

3. (Hard step!) rewrite the right hand side in terms of  $g(x)$  and/or other functions. Usually done by shifting sums, multiplying series, and identifying common series.
4. Now solve for  $g(x)$ , and then expand this into a series to read off the coefficient to get  $a_n$ . For example, this can be done using general binomial theorem or partial fractions.

The best way to see this is through lots of examples.

*Example:* Use a generating function to solve the recurrence  $a_n = a_{n-1} + 2a_{n-2} + 1$  with  $a_0 = 1$  and  $a_1 = 2$ .

*Solution:* Following the procedure for finding  $g(x)$  given above we have

$$\begin{aligned} g(x) &= \sum_{n \geq 0} a_n x^n \\ &= a_0 + a_1 x + \sum_{n \geq 2} a_n x^n \\ &= 1 + 2x + \sum_{n \geq 2} (a_{n-1} + 2a_{n-2} + 1) x^n \\ &= 1 + 2x + \sum_{n \geq 2} a_{n-1} x^n + 2 \sum_{n \geq 2} a_{n-2} x^n + \sum_{n \geq 2} x^n \\ &= 1 + 2x + x(g(x) - 1) + 2x^2 g(x) + \frac{x^2}{1-x}. \end{aligned}$$

Before continuing we should see how the last step happened. We have

$$\begin{aligned} \sum_{n \geq 2} a_{n-1} x^n &= a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots \\ &= x(a_1 x + a_2 x^2 + a_3 x^3 + \dots) = x(g(x) - a_0), \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{n \geq 2} a_{n-2} x^n &= a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots \\ &= x^2(a_0 + a_1 x + a_2 x^2 + \dots) = x^2 g(x). \end{aligned}$$

(This can also be done by factoring out an  $x^2$  and then shifting the index.) Finally since  $1 + x + x^2 + \dots = 1/(1-x)$  then

$$\sum_{n \geq 2} x^n = x^2 \sum_{n \geq 2} x^{n-2} = x^2 \frac{1}{1-x}.$$

We now solve for  $g(x)$ , doing that we get the following.

$$(1 - x - 2x^2)g(x) = 1 + x + \frac{x^2}{1-x} = \frac{1}{1-x}.$$

Dividing we finally have the generating function

$$g(x) = \frac{1}{(1-x)(1-x-2x^2)} = \frac{1}{(1-x)(1+x)(1-2x)}.$$

We now want to break this up, which we can do using the techniques of partial fractions,

$$\frac{1}{(1-x)(1+x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-2x}.$$

Clearing denominators this becomes

$$1 = A(1+x)(1-2x) + B(1-x)(1-2x) + C(1+x)(1-x).$$

We can now expand and collect the coefficient of powers of  $x$  and set up a system of equations (which works great), but before we do that let us observe that this must be true for all values of  $x$ . So let us choose some "nice" values of  $x$ , namely values where most of the terms drop out. So for instance we have

$$\begin{aligned} x = 1 & \text{ becomes } 1 = -2A \text{ giving } A = -\frac{1}{2}, \\ x = -1 & \text{ becomes } 1 = 6B \text{ giving } B = \frac{1}{6}, \\ x = \frac{1}{2} & \text{ becomes } 1 = \frac{3}{4}C \text{ giving } C = \frac{4}{3}. \end{aligned}$$

The final ingredient will be using

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots,$$

to get

$$\begin{aligned} g(x) &= -\frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{6} \cdot \frac{1}{1+x} + \frac{4}{3} \cdot \frac{1}{1-2x} \\ &= -\frac{1}{2} \sum_{n \geq 0} x^n + \frac{1}{6} \sum_{n \geq 0} (-x)^n + \frac{4}{3} \sum_{n \geq 0} (2x)^n \\ &= \sum_{n \geq 0} \left( -\frac{1}{2} + \frac{1}{6}(-1)^n + \frac{4}{3}2^n \right) x^n. \end{aligned}$$

So we have that

$$a_n = -\frac{1}{2} + \frac{1}{6}(-1)^n + \frac{4}{3}2^n.$$

*Example:* Let  $f(x) = \sum_{n \geq 0} F_n x^n$  where  $F_n$  are the Fibonacci numbers ( $F_0 = 0$ ,  $F_1 = 1$  and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ ). Find a simple expression for  $f(x)$ .

*Solution:* We proceed as before. We have

$$\begin{aligned} f(x) &= \sum_{n \geq 0} F_n x^n \\ &= F_0 + F_1 x + \sum_{n \geq 2} (F_{n-1} + F_{n-2}) x^n \\ &= x + x \sum_{n \geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} F_{n-2} x^{n-2} \\ &= x + x f(x) + x^2 f(x). \end{aligned}$$

So we have  $(1 - x - x^2)f(x) = x$  or

$$f(x) = \frac{x}{1 - x - x^2}.$$

We can actually use this to find an expression for the Fibonacci numbers (similar to what we got previously). To do this we will let  $\phi = (1 + \sqrt{5})/2$  and  $\hat{\phi} = (1 - \sqrt{5})/2$  then it can be checked that

$$(1 - x - x^2) = (1 - \phi x)(1 - \hat{\phi} x).$$

We now have

$$\frac{x}{1 - x - x^2} = \frac{A}{1 - \phi x} + \frac{B}{1 - \hat{\phi} x}.$$

clearing denominators we have

$$x = A(1 - \hat{\phi} x) + B(1 - \phi x) = (A + B) - (A\hat{\phi} + B\phi)x$$

We can conclude that  $A = -B$  and that

$$-1 = A\hat{\phi} + B\phi = B(\phi - \hat{\phi}) = \sqrt{5}B.$$

So  $B = -1/\sqrt{5}$  and  $A = 1/\sqrt{5}$ , giving

$$\begin{aligned} f(x) &= \frac{x}{1 - x - x^2} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x} \\ &= \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\phi x)^n - \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\hat{\phi} x)^n \\ &= \sum_{n \geq 0} \left( \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \hat{\phi}^n \right) x^n, \end{aligned}$$

or substituting for  $\phi$  and  $\hat{\phi}$  we get

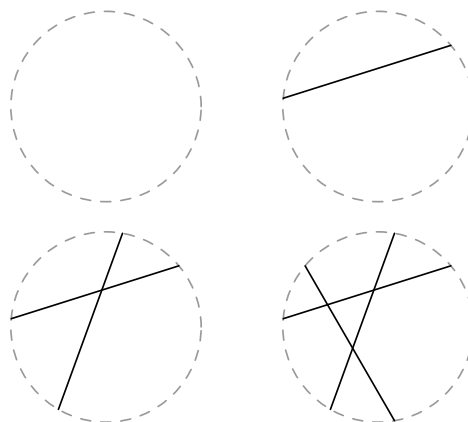
$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n,$$

as we have found previously.

(To be careful we should point out that what we have done only makes sense if our function is analytic around some value of  $x$  near 0. In our case the nearest pole for the function  $f(x)$  is at  $x = \hat{\phi}$  so that near  $x = 0$  everything that we have done is fine. We will not worry about such matters in our course, but if you are interested in finding more about this topic read *Analytic Combinatorics* by Flajolet and Sedgewick.)

*Example:* Let  $a_n$  be the number of regions that  $n$  lines divide the plane into. (Assume that no two lines are parallel and no three meet at a point.) Solve for  $a_n$  using generating functions.

*Solution:* Let us start by looking at a few simple cases. From the pictures below we see that  $a_0 = 1$ ,  $a_1 = 2$ ,  $a_2 = 4$  and  $a_3 = 7$ .



In general when we draw in the  $n$ th line we start drawing it in from  $\infty$ , every time we cross one of the  $n - 1$  lines we will create one additional region, and then as we head to  $\infty$  we will create one last region. So we have the recurrence

$$a_n = a_{n-1} + n,$$

with initial condition  $a_0 = 1$ . (Checking we see that this gives the right answer for  $a_1$ ,  $a_2$ ,  $a_3$  and  $a_4$ , which is good!)

Now we have

$$\begin{aligned} g(x) &= \sum_{n \geq 0} a_n x^n \\ &= a_0 + \sum_{n \geq 1} (a_{n-1} + n) x^n \\ &= 1 + x \sum_{n \geq 1} a_{n-1} x^{n-1} + \sum_{n \geq 1} n x^n \\ &= 1 + x g(x) + \sum_{n \geq 1} n x^n. \end{aligned}$$

The hard part about this is the second part of the sum. To handle we start with

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n.$$

Taking the derivative of both sides we get

$$\frac{1}{(1-x)^2} = \sum_{n \geq 0} nx^{n-1},$$

almost what we want, we just multiply by  $x$  to get

$$\frac{x}{(1-x)^2} = \sum_{n \geq 0} nx^n.$$

(Note of course that it doesn't matter whether we start our sum at 0 or 1, the result is the same.) So we have

$$(1-x)g(x) = 1 + \frac{x}{(1-x)^2},$$

or

$$\begin{aligned} g(x) &= \frac{1}{1-x} + \frac{1-(1-x)}{(1-x)^3} \\ &= \frac{1}{1-x} - \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3}. \end{aligned}$$

From when we started using generating functions we saw that

$$\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{n} x^n,$$

and so we have

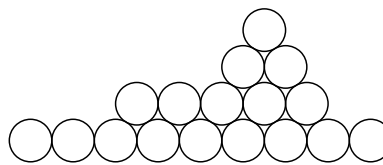
$$\begin{aligned} g(x) &= \sum_{n \geq 0} x^n - \sum_{n \geq 0} \binom{n+1}{n} x^n + \sum_{n \geq 0} \binom{n+2}{n} x^n \\ &= \sum_{n \geq 0} \left( 1 - (n+1) + \frac{(n+2)(n+1)}{2} \right) x^n \\ &= \sum_{n \geq 0} \left( \frac{n^2 + n + 2}{2} \right) x^n. \end{aligned}$$

So we can conclude that  $a_n = \frac{n^2 + n + 2}{2}$ .

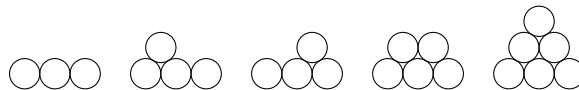
So far we have looked at examples that we could already solve using other methods. Let us try an example that we could not solve (or at least would be very hard!) with our previous methods. The following example is taken from the book *generatingfunctionology*.

*Example:* A block fountain of coins is an arrangement of coins in rows such that each row of coins forms a single contiguous block and each coin in the second or higher row touches two coins below (an example is shown below). Let  $f(n)$  denote the number of block

fountains with  $n$  coins in the bottom row. Find the generating function for  $f(n)$ .



*Solution:* We let  $f(0) = 1$  (corresponding to the empty configuration). Clearly we have  $f(1) = 1$  and  $f(2) = 2$ . The case  $f(3) = 5$  is shown below.



Our next step is to find a recurrence. The fountain block either consists only of the single bottom row (one configuration) or it has another fountain block stacked on top of the bottom row. Suppose that the bottom row has length  $n$  and the second row has length  $j$ , then the second row can go in any of  $n - j$  positions. Putting it altogether we have

$$f(n) = 1 + \sum_{j=1}^n (n-j)f(j).$$

(The 1 corresponds to the single row solution, the sum corresponds to when we have multiple rows. Note that when  $j = n$  the contribution will be 0 in the sum.)

So we have

$$\begin{aligned} g(x) &= \sum_{n \geq 0} f(n)x^n \\ &= f(0) + \sum_{n \geq 1} \left( 1 + \sum_{j=1}^n (n-j)f(j) \right) x^n \\ &= 1 + \sum_{n \geq 1} x^n + \sum_{n \geq 1} \left( \sum_{j=1}^n (n-j)f(j) \right) x^n \\ &= 1 + \frac{x}{1-x} + \sum_{n \geq 1} \left( \sum_{j=1}^n (n-j)f(j) \right) x^n. \end{aligned}$$

The difficult step is determining what to do with

$$\sum_{n \geq 1} \left( \sum_{j=1}^n (n-j)f(j) \right) x^n,$$

looking at it the inside reminds us of  $a_0b_n + a_1b_{n-1} + \dots + a_nb_0$ , i.e., where we multiply two series together. It is not too hard to check that

$$\begin{aligned} &\sum_{n \geq 1} \left( \sum_{j=1}^n (n-j)f(j) \right) x^n \\ &= (x + 2x^2 + 3x^3 + \dots)(f(1)x + f(2)x^2 + f(3)x^3 \dots) \\ &= \frac{x}{(1-x)^2}(g(x) - 1). \end{aligned}$$

So combining we have

$$g(x) = 1 + \frac{x}{1-x} + \frac{x}{(1-x)^2}(g(x) - 1),$$

or

$$\left(1 - \frac{x}{(1-x)^2}\right)g(x) = 1 + \frac{x}{1-x} - \frac{x}{(1-x)^2}.$$

Multiplying both sides by  $(1-x)^2$  and simplifying we have

$$(1 - 3x + x^2)g(x) = 1 - 2x \quad \text{or} \quad g(x) = \frac{1 - 2x}{1 - 3x + x^2}.$$