Lecture 6 – April 10

As another application of the binomial theorem we have the following.

> For $n \geq 1$
> $$\sum_{k=0}^{n} (-1)^n \binom{n}{k} = 0.$$

In other words starting with the second row and going down if we sum along the rows alternating sign as we go the result is 0. Given the symmetry $\binom{n}{k} = \binom{n}{n-k}$ it trivially holds when k is odd, it is not trivial to show that it holds when k is even.

Algebraic proof: In the binomial theorem set $x = -1$ and $y = 1$ to get

$$0 = (-1+1)^n = \sum_{k=0}^{n} (-1)^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^{n} 1^k \binom{n}{k}. \quad \square$$

Combinatorial proof: First note that this is equivalent to showing the following:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

The left hand side counts the number of subsets of $\{1, 2, \ldots, n\}$ with an *even* number of elements while the right hand side counts the number of subsets with an *odd* number of elements. To show that we have an equal number of these two types we will use an *involution argument*.

An involution on a set X is a mapping $\phi : X \to X$ so that $\phi(\phi(x)) = x$. Given an involution ϕ the elements then naturally split into fixed points (elements with $\phi(x) = x$) or pairs (two elements $x \neq y$ where $\phi(x) = y$ and $\phi(y) = x$).

In our case our involution will act on $2^{[n]}$ (the set of all subsets of $\{1, 2, \ldots, n\}$) and is defined for a subset A as follows:

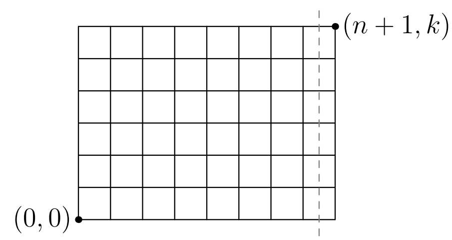
$$\phi(A) = \begin{cases} A \setminus \{1\} & \text{if 1 is in } A, \\ A \cup \{1\} & \text{if 1 is not in } A. \end{cases}$$

In other words we take a set and if 1 is in it we remove it and if 1 is not in it we add it. We now make some observations. First, for any subset A we have $\phi(\phi(A)) = A$ so that it is an involution. Further, there are no fixed points since 1 must either be in or not in a set. So we can now break the collection of subsets into pairs $\{A, \phi(A)\}$. Finally, by the involution A and $\phi(A)$ will differ by exactly one element so one of them is even and one of them is odd. So we now have a way to pair every subset with an even number of elements with a subset with an odd number of elements. So the number of such subsets must be equal. $\square$

We now give two identities which are useful for simplifying sums of binomial coefficients.

> $$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k},$$
> and
> $$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

To prove the first one we will count the number of walks from $(0, 0)$ to $(n+1, k)$ using right steps and up steps in two ways.



We must make $n + 1$ steps to the right and k steps up. So the number of ways to walk from one corner to the other is the number of ways to choose when to make the up steps which is

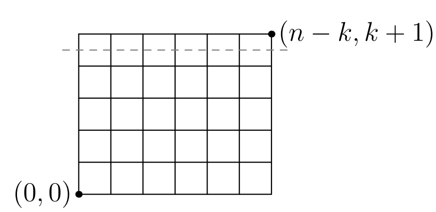
$$\binom{n+k+1}{k}.$$

We can also count the number of walks by grouping them according to which line segment is used in the last step to the right (in the picture above this corresponds to grouping according to the line segment which intersects the dotted line). These line segments go from (n, i) to $(n + 1, i)$ where $i = 0, \ldots, k$. Once we have crossed the line segment there is only one way to finish the walk to the corner (straight up the side). On the other hand the number of ways to get to (n, i) is $\binom{n+i}{i}$. So by the rule of addition we have that the total number of walks is

$$\binom{n+0}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k}.$$

Combining these two different ways to count the paths gives the identity.

A proof of the other result can be done similarly using the following picture (we leave it to the interested reader to fill in the details).



These identities can also be proved by using $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. For example, we have the following.

$$\begin{aligned}\binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \\ &= \binom{n}{k} + \binom{n-1}{k} + \binom{n-1}{k+1} \\ &= \cdots \\ &= \binom{n}{k}+\binom{n-1}{k}+\cdots+\binom{k+1}{k}+\binom{k+1}{k+1} \\ &= \binom{n}{k}+\binom{n-1}{k}+\cdots+\binom{k+1}{k}+\binom{k}{k}.\end{aligned}$$

When I was taught these identities they were called the "hockey stick" identities. This name comes from the pattern that they form in Pascal's triangle. For instance we have that $\binom{6}{3}$ (shown below in blue) is the $\binom{2}{2} + \cdots + \binom{5}{2}$ (shown below in red).

$$\begin{array}{c} \binom{0}{0} \\ \binom{1}{0}\ \binom{1}{1} \\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\ \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\ \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\ \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \end{array}$$

The other identity corresponds to looking at the mirror image of Pascal's triangle.

We now give an application of the hockey stick identity.

$$\boxed{\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

In the first lecture we saw how to add up the sum of i by counting dots in two different ways. Here we want to sum up i^2, the problem is that we don't have an easy method to do that. We *do* have an easy method to add up the binomial coefficients, so if we can rewrite i^2 in terms of binomial coefficients then we can easily answer this question. So consider the following.

$$i^2 = 2\frac{i(i-1)}{2} + i = 2\binom{i}{2} + \binom{i}{1}.$$

(The ability to rewrite i^2 as a combination of binomial coefficients can also be used for any other polynomial expression of i. The trick is to find the coefficients involved. For the case of polynomials of the form i^ℓ we have that the coefficient of $\binom{i}{k}$ is $k!\left\{ {\ell \atop k} \right\}$ where $\left\{ {\ell \atop k} \right\}$ are the Stirling numbers of the second kind, which we will discuss in a later lecture.)

Now using this new way to write i^2 we have

$$\begin{aligned}\sum_{i=0}^{n} i^2 &= \sum_{i=0}^{n}\left(2\binom{i}{2} + \binom{i}{1}\right) \\ &= 2\sum_{i=0}^{n}\binom{i}{2} + \sum_{i=0}^{n}\binom{i}{1} \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\ &= \frac{(n+1)n}{6}\big(2(n-1)+3\big) \\ &= \frac{(n+1)n(2n+1)}{6}.\end{aligned}$$

The only difficult step now is going from the second to the third line, which is done using the hockey stick identity.

There is a more general form of the binomial theorem known as the multinomial theorem ("multinomial" referring to many variables as compared to "binomial" which refers to two). It states

$$\begin{aligned}&(x_1 + x_2 + \cdots + x_k)^n \\ &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1\ge0,n_2\ge0,\ldots,n_k\ge0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},\end{aligned}$$

where

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

is the multinomial coefficient we have seen in previous lectures.

Returning to the binomial theorem, we have that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.$$

In particular for n fixed all of the binomial coefficients $\binom{n}{k}$ are "stored" as coefficients inside of the function $f(x) = (1+x)^n$. In general given any sequence of numbers $a_0, a_1, a_2, \ldots$ (possibly infinitely many) we can store these as coefficients of a function which we call the *generating function* as follows

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

(There are different ways that we can store the sequence as coefficients leading to different classes of generating functions. This method is what is known as an ordinary generating function, in a later lecture we will see what are known as exponential generating functions.)

Generally speaking we will treat the function as a "formal" series, i.e., as an algebraic rather than analytic structure. This makes it convenient to work with when we do not need to worry about convergence. Nevertheless we will make extensive use of analytic techniques, when we do that is when we will start to worry about convergence.

Let us now consider the problem of finding the generating function for the sequence $a_n = \binom{n}{k}$ where k is fixed. In other words we want to find a function so that its series expansion (note that there are infinitely many a_n in this case) is

$$g(y) = \sum_n \binom{n}{k} y^n.$$

We can do this by actually investigating a much more general function. Namely we will use a generating function with two variables x and y and is defined as follows:

$$h(x,y) = \sum_{k,n} \binom{n}{k} x^k y^n.$$

Using the binomial theorem we have

$$\begin{aligned} h(x,y) &= \sum_{k,n} \binom{n}{k} x^k y^n \\ &= \sum_n y^n \Big(\sum_k \binom{n}{k} x^k \Big) \\ &= \sum_n y^n (1+x)^n = \sum_n \big(y(1+x)\big)^n \\ &= \frac{1}{1-y(1+x)} = \frac{1}{1-y-xy}. \end{aligned}$$

(Note that this function stores the *entire* Pascal's triangle inside of it.)

In the above derivation we used the following important identity,

$$1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Analytically this is true for $|z| < 1$, formally this is always true.

Now to find our desired $g(y)$ we are looking for the coefficient of x^k. So we now rewrite $h(x,y)$ as a power series in x and get

$$\begin{aligned} h(x,y) &= \frac{1}{1-y-xy} = \frac{1}{1-y} \frac{1}{1-\frac{y}{1-y}x} \\ &= \frac{1}{1-y} \sum_{k=0}^{\infty} \Big(\frac{y}{1-y} x \Big)^k \\ &= \sum_{k=0}^{\infty} \Big(\frac{y^k}{(1-y)^{k+1}} \Big) x^k. \end{aligned}$$

Finally, reading off the coefficient for x^k in $h(x,y)$ gives us our desired function, namely

$$g(y) = \sum_n \binom{n}{k} y^n = \frac{y^k}{(1-y)^{k+1}}.$$

Lecture 7 – April 13

Today we look at one motivation for studying generating functions, namely a connection between polynomials and distribution problems. Let us start with a simple example.

Example: Give an interpretation for the coefficient of $x^1 1$ for the polynomial

$$g(x) = (1 + x^2 + x^3 + x^5)^3.$$

Solution: First note that we can write $g(x)$ as

$$(1 + x^2 + x^3 + x^5)(1 + x^2 + x^3 + x^5)(1 + x^2 + x^3 + x^5)$$

and that a typical term which we will get when we expand has the form $x^{n_1} x^{n_2} x^{n_3}$ where the term x^{n_1} comes from picking a term from the first polynomial (so $n_1 \in \{0,2,3,5\}$), the term x^{n_2} comes from picking a term from the second polynomial (so $n_2 \in \{0,2,3,5\}$), and the term x^{n_3} comes from picking a term from the third polynomial (so $n_3 \in \{0,2,3,5\}$). Since we are interested in the coefficient of x^{11} then we need $n_1 + n_2 + n_3 = 11$.

So this can be rephrased as a balls and bins problem where we are trying to distribute 11 balls among three bins where in each bin we can have either 0, 2, 3 or 5 balls.

Example: Give an interpretation for the coefficient of x^r for the function

$$g(x) = (1+x+x^2)(1+x+x^2+\cdots)(x+x^3+x^5+\cdots).$$

Solution: By the same reasoning as we have before a typical term in the expansion looks like $x^{n_1+n_2+n_3}$ with n_1 being 0, 1 or 2; $n_2 \geq 0$; and $n_3 \geq 0$ and even.

So this can be rephrased as a balls and bins problem where we are trying to distribute r balls among three bins where in the first bin we put 0, 1 or 2 balls; in the second bin we put any number of balls we want; in the third bin we put in an odd number of balls.

In general given a polynomial

$$g(x) = g_1(x) g_2(x) \cdots g_k(x)$$

with each g_i a sum of some power of xs, i.e.,

$$g_i(x) = \sum_\ell x^{q_\ell^i}.$$

Then the coefficient of x^r in $g(x)$ is the number of ways to place r balls into k bins so that in the ith bin the number of balls is q_ℓ^i for some ℓ. (The idea is that the term $g_i(x)$ tells you the restrictions about the kind of balls that can be placed into that bin. It is important that the coefficients of the $g_i(x)$ are all 1's.)

We can also start with a balls and bins problem and translate it into finding the coefficient of some appropriate polynomial.

Example: Translate the following problem into finding a coefficient of some function: "How many ways are there to put 12 balls into four bins so that the first two bins have between 2 and 5 balls, the third bin has at least 3 balls and the last bin has an odd number of balls?"

Solution: Since there will be 12 balls we will be looking for the coefficient of x^{12}. Now let us translate the condition on each bin. For the first two bins there are between 2 and 5 balls so

$$g_1(x) = g_2(x) = x^2 + x^3 + x^4 + x^5$$

(remember, the powers list the number of balls that can be put in the bin). For the third bin we have to have at least 3 balls so

$$g_3(x) = x^3 + x^4 + \cdots .$$

We could also use the function

$$g_3'(x) = x^3 + x^4 + \cdots + x^{12},$$

the difference being that we stop and have a polynomial instead of an infinite series. The reason we can do this is that we are only interested in the coefficient of x^{12}, anything which will not contribute to that coefficient can be dropped without changing the answer. However by keeping the series we can use the same function to answer the question for any arbitrary number of balls and so the first option is more general. Finally for the last bin we have

$$g_4(x) = x + x^3 + x^5 + \cdots .$$

Putting it altogether we are trying to find the coefficient of x^{12} for the function

$$(x^2+x^3+x^4+x^5)^2(x^3+x^4+\cdots)(x+x^3+x^5+\cdots).$$

Of course translating one problem to another is only good if we have a technique to solve the second problem. So our next natural step is to find ways to determine the coefficient of functions of this type. To help us do this we will make use of the following identities.

$$\begin{aligned}
(1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \\
(1-x^m)^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{mk} \\
\frac{1-x^{m+1}}{1-x} &= 1 + x + x^2 + \cdots + x^m \\
\frac{1}{1-x} &= 1 + x + x^2 + \cdots \qquad (\text{for } |x|<1) \\
\frac{1}{(1-x)^n} &= \sum_{k\ge 0} \binom{n-1+k}{k} x^k
\end{aligned}$$

The first two are simple applications of the binomial theorem. The third is easily verifiable by multiplying both sides by $(1-x)$ and simplifying. The fourth one is a well known sum and can be considered the limiting case of the third one. (Note that as a function of x, i.e., analytically, this makes sense only when $|x|<1$. Formally there is no restriction on x, this is because formally x^k is acting as a placeholder.)

To see why the last one holds we note that this is

$$\begin{aligned}
\frac{1}{(1-x)^n} &= \underbrace{\frac{1}{1-x}\frac{1}{1-x}\cdots\frac{1}{1-x}}_{n \text{ times}} \\
&= \underbrace{(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n \text{ times}}
\end{aligned}$$

Translating this into a balls and bins problem, the coefficient of x^k would correspond to the number of solutions of

$$e_1 + e_2 + \cdots + e_n = k$$

where each $e_i \ge 0$. We have already solved this problem using bars and stars, namely this can be done in $\binom{n-1+k}{k}$ ways, giving us the desired result.

We will need one more tool, and that is a way to multiply functions. We have the following which is a simple exercise in expanding and grouping coefficients.

Given

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots , \\
g(x) &= b_0 + b_1 x + b_2 x^2 + \cdots .
\end{aligned}$$

Then

$$\begin{aligned}
f(x)g(x) &= a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots \\
&\quad + (a_0b_k + a_1b_{k-1} + \cdots + a_kb_0)x^k + \cdots .
\end{aligned}$$

The key is that the coefficient of x^k is found by combining elements $(a_i x^i)(b_{k-i}x^{k-1})$ for $i = 0, \ldots, k$.

We now show how to use these various rules to solve a combinatorial problem.

Example: How many ways are there to put 25 balls into 7 boxes so that each box has between 2 and 6 balls?

Solution: First we translate this into a polynomial problem. We are looking for 25 balls so we will be looking for a coefficient of x^{25}. The constraint on each box is the same, namely the between 2 and 6 balls so that the function we will be considering is

$$\begin{aligned} g(x) &= (x^2 + x^3 + x^4 + x^5 + x^6)^7 \\ &= x^{14}(1 + x + x^2 + x^3 + x^4)^7. \end{aligned}$$

Here we pulled out an x^2 out of the inside which then becomes x^{14} in front. Now since we are looking for the coefficient of x^{25} and we have a factor of x^{14} in front, our problem is equivalent to finding the coefficient of x^{11} in

$$g'(x) = (1 + x + x^2 + x^3 + x^4)^7.$$

(Combinatorially, this is also what we would do. Namely distribute two balls into each bin, fourteen balls in total, and then decide how to deal with the remaining 11.)

Using our identities we have

$$g'(x) = (1 + x + x^2 + x^3 + x^4)^7$$

$$= \left(\frac{1-x^5}{1-x}\right)^7 = (1-x^5)^7 \cdot \frac{1}{(1-x)^7}$$

$$= \left(\binom{7}{0} - \binom{7}{1}x^5 + \binom{7}{2}x^{10} + \cdots\right)\left(\sum_{k\ge 0}\binom{6+k}{k}x^k\right)$$

Finally, we use the rule for multiplying functions together to find the coefficient of x^{11}. In particular note that in the first part of the product only three terms can be used, as all the rest will not contribute to the coefficient of x^{11}. So multiplying we have that the coefficient to x^{11} is

$$\binom{7}{0}\binom{6+11}{11} - \binom{7}{1}\binom{6+6}{6} + \binom{7}{2}\binom{6+1}{1} = 6{,}055.$$

This can also be done combinatorially. The first term counts the numbers without restriction. The second term then takes off the exceptional cases, but it takes off too much and so the third term is there to add it back in.

Lecture 8 – April 15

We can use the rule for multiplying functions together to prove various results. For instance, let $f(x) = (1+x)^m$ and $g(x) = (1+x)^n$, then $f(x)g(x) = (1+x)^{m+n}$. We now compute the coefficient of x^r of $f(x)g(x)$ in two different ways to get

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

The left hand side follows by using the rule of multiplying $f(x)$ and $g(x)$ while the right hand side is the binomial theorem for $f(x)g(x)$.

We turn to a problem of finding a generating function.

Example: Show that the generating function for the number of integer solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

with $0 \le e_1 \le e_2 \le e_3 \le e_4$ is

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

Solution: First note that we can rewrite the function $f(x)$ as

$$f(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots) \times (1+x^3+x^6+x^9+\cdots)(1+x^4+x^8+x^{12}+\cdots).$$

Note that translating this back into a balls and bins problem this says that we have four bins. In the first bin we have any number of balls, in the second bin we have a multiple of two number of balls, in the third bin we have a multiple of three number of balls and in the fourth bin we have a multiple of four number of balls. In other words the generating function $f(x)$ counts the number of solutions to

$$f_1 + 2f_2 + 3f_3 + 4f_4 = n$$

with $f_1, f_2, f_3, f_4 \ge 0$. We now need to show that these two problems have the same number of solutions. To do this, let us start with a solution of $e_1+e_2+e_3+e_4 = n$ and pictorially represent our solution by putting four row of $*$s with e_4 stars in the first row, e_3 stars in the second row, e_2 stars in the third row and e_1 stars in the fourth row. Finally, let f_4 be the number of columns with four $*$s, f_3 the number of columns with three $*$s, f_2 the number of columns with two $*$s and f_1 the number of columns with one $*$. This gives a solution to $f_1 + 2f_2 + 3f_3 + 4f_4 = n$. This gives a one-to-one correspondence between solutions to the two different problems, so they have an equal number of solutions giving the result.

As an example of the last step, suppose we start with

$$3 + 5 + 9 + 11 = 28.$$

Then pictorially this corresponds to the following picture (we have marked the number of $*$s in each row/column).

11	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
9	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$		
5	$*$	$*$	$*$	$*$	$*$						
3	$*$	$*$	$*$								
	4	4	4	3	3	2	2	2	2	1	1

So translating this gives us the solution

$$(2) + 2(4) + 3(2) + 4(3) = 28$$

to the second problem.

This problem says that the number of ways to break n into a sum of at most four parts is the same as the number of ways to break n into a sum of parts where each part has size at most four. We will generalize this idea in the next lecture. For now we start by looking at partitions. A partition of a number n is a way to break n up as a sum of smaller pieces. For instance

$$1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 3 = 2 + 2 = 4$$

are the ways to break four into pieces. We do not care about the order of the pieces so that $1 + 1 + 2$ and $1 + 2 + 1$ and $2 + 1 + 1$ are considered the same partition. This corresponds to the number of ways to distribute identical balls among identical bins (for now we will suppose that we have an unlimited number of bins so that we can have any number of parts in the partition).

Let $p(n)$ denote the number of partitions of n.

n	partitions of n	$p(n)$
0	0	1
1	1	1
2	$1+1$, 2	2
3	$1+1+1$, $1+2$, 3	3
4	$1+1+1+1$, $1+1+2$, $1+3$ $2+2$, 4	5
5	$1+1+1+1+1$, $1+1+1+2$ $1+1+3$, $1+2+2$, $1+4$ $2+3$, 5	7
6	$1+1+1+1+1+1$ $1+1+1+1+2$, $1+1+1+3$ $1+1+2+2$, $1+1+4$, $1+2+3$ $1+5$, $2+2+2$, $2+4$, $3+3$, 6	11

The function $p(n)$ has been well studied, but is highly non-trivial. One of the greatest mathematical geniuses of the twentieth century was the Indian mathematician Srinivasa Ramanujan. Originally a clerk in India he was able through personal study discover dozens of previously unknown relationships about partitions. He sent these (along with other discoveries) to mathematicians in England and one of them, G. H. Hardy, was able to recognize the writings as something new and exciting which caused him to bring Ramanujan to England which resulted in one of the great mathematical collaborations of the twentieth century. Examples of facts that Ramanujan discovered include that $p(5n + 4)$ is divisible by 5; $p(7n + 5)$ is divisible by 7; and $p(11n + 6)$ is divisible by 11.

One natural question is to whether the exact value of $p(n)$ is known. There is a "nice" formula for $p(n)$, namely

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\left[\frac{d}{dx}\frac{\sinh\left(\frac{\pi}{k}\sqrt{x-\frac{1}{24}}\right)}{\sqrt{x-\frac{1}{24}}}\right]_{x=n}$$

where $A_k(n)$ is a specific sum of $24k$th roots of unity. Proving this is beyond the scope of our class!

We now turn to the (much) easier problem of finding the generating function for $p(n)$, that is we want to find

$$\sum_{n\geq 0} p(n)x^n.$$

The first thing is to observe that a partition of size n corresponds to a solution of

$$e_1 + 2e_2 + 3e_3 + \cdots + ke_k + \cdots = n,$$

where each e_i represents the number of parts of size i. Notice in particular the the contribution of e_k will be a multiple of k. So turning this back into balls and bins we have n balls and infinitely many bins. So the number of solutions (using what we did last time) is

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \underbrace{(1+x+x^2+x^3+\cdots)}_{e_1\text{ term}} \\ &\times \underbrace{(1+x^2+x^4+x^6+\cdots)}_{e_2\text{ term}} \times \underbrace{(1+x^3+x^6+x^9+\cdots)}_{e_3\text{ term}} \\ &\times \cdots \times \underbrace{(1+x^k+x^{2k}+x^{3k}+\cdots)}_{e_k\text{ term}} \times \cdots .\end{aligned}$$

Since

$$1 + z^k + z^{2k} + z^{3k} + \cdots = \frac{1}{1-z^k},$$

we can rewrite the generating function as

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots\frac{1}{1-x^k}\cdots \\ &= \prod_{k\geq 1}\frac{1}{1-x^k}.\end{aligned}$$

We can do variations on this. For instance we can look at partitions of n where no part is repeated. In other words we want to restrict our partitions so that $e_k = 0$ or 1 for each k. If we denote these partitions by $p_d(n)$ then

$$\begin{aligned}\sum_{n\geq 0} p_d(n)x^n &= (1+x)(1+x^2)\cdots(1+x^k)\cdots \\ &= \prod_{k\geq 1}(1+x^k)\end{aligned}$$

Similarly we can count partitions with each part odd. In other words we want to restrict out partitions so that $e_{2k} = 0$ for each k. If we denote these partitions by $p_o(n)$ then it is easy to modify our construction for $p(n)$ to get

$$\sum_{n\geq 0} p_o(n) = \prod_{k\geq 1}\frac{1}{1-x^{2k-1}}.$$

We can combine these two generating functions to derive a remarkable result.

For $n \geq 1$ we have $p_d(n) = p_o(n)$.

Remarkably, we do not know in general what $p_d(n)$ or $p_o(n)$ is, nevertheless we do know that they are equal! As an example of this we have that $p_o(6) = 4$ because of the partitions

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3,$$

while $p_d(6) = 4$ because of the partitions

$$1+2+3, \quad 1+5, \quad 2+4, \quad 6$$

It suffices for us to show that the two generating functions are identical. If the functions are the same then the coefficients must be the same giving us the desired result. So we have that

$$\begin{aligned}\sum_{n\geq 0} p_d(n)x^n &= \prod_{k\geq 1}(1+x^k) \\ &= \prod_{k\geq 1}\frac{1-x^{2k}}{1-x^k} \\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^3}\cdots \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots \\ &= \prod_{k\geq 1}\frac{1}{1-x^{2k-1}} = \sum_{n\geq 0} p_o(n)x^n.\end{aligned}$$

We can also give a bijective proof. To do this we need to give a way to take a partition with odd parts and produce a (unique) partition with distinct parts, and vice versa. This can be done by repeatedly applying the following rule until it cannot be done anymore:

- With the current partition, if any two parts are equal combine them into a single part.

For example for the partition

$$\begin{aligned}1+1+1+1+1+1 &\to 2+1+1+1+1 \\ &\to 2+2+1+1 \\ &\to 4+1+1 \\ &\to 4+2\end{aligned}$$

This rule takes a partition with odd parts and produces a partition with distinct parts. To go in the opposite direction this cane be done by applying the following rule until it cannot be done anymore:

- With the current partition, if any part is even then split it into two equal parts.

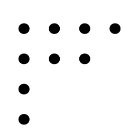
For example for the partition

$$\begin{aligned}3+4+6 &\to 2+2+3+6 \\ &\to 1+1+2+3+6 \\ &\to 1+1+1+1+3+6 \\ &\to 1+1+1+1+3+3+3\end{aligned}$$

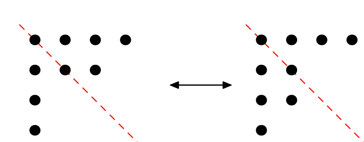
These two rules give a bijective relationship between partitions with an odd number of parts and partitions with distinct parts.

Lecture 9 – April 17

We can visually represent a partition as a series of rows of dots (much like we did in the example in the previous lecture). This representation is known as a *Ferrer's diagram*. The number of dots in each row is the size of the part and we arrange the rows in weakly decreasing order. The partition $4+3+1+1$ would be represented by the following diagram.

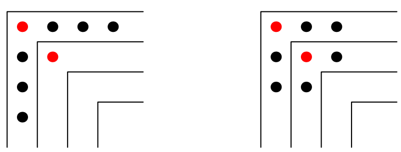


We can use Ferrer's diagrams to gain insight into properties of partitions. For instance one simple operation that we can do is to take the transpose (or the conjugate) of the partition. This is done by "flipping" across the line in the picture below, so that the columns become rows and rows become columns.



Since we do not change the number of dots by taking the transpose this takes a partition and makes a new partition. So in this case we get the new partition $4+2+2+1$.

Note that if we take the transpose of the transpose that we get back the original diagram. There are some partitions such that the transpose of the partition gives back the partition, such partitions are called self conjugate. For instance there are two self conjugate partitions for $n=8$, namely $4+2+1+1$ and $3+3+2$. A famous result says that the number of self conjugate partitions is equal to the number of partitions with distinct odd parts. So for example for $n=8$ there are two partitions with distinct odd parts, namely $7+1$ and $5+3$. One proof of this relationship is based on using Ferrer's diagrams. We will not fill in all of the details here but give the following "hint" for $n=8$.



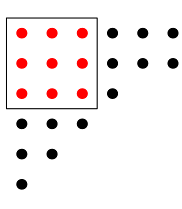
Also note that when we take the transposition that the number of rows becomes the size of the largest part in the transposition while the size of the largest part becomes the number of rows. So mapping a partition to its transpose establishes the following fact.

> There is a one-to-one correspondence between the number of partitions of n into exactly m parts and the number of partitions of n into parts of size at most m with at least one part of size m.

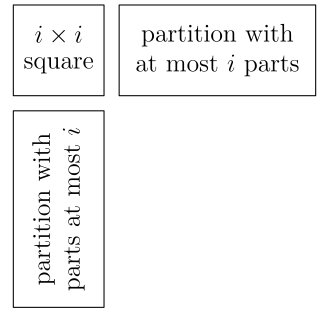
It is easy to count partitions with parts of size at most m with at least one part of size m. In particular if we let $\left|{n \atop m}\right|$ denote the number of partitions of n into exactly m parts then we have (using the techniques from the last lecture)

$$\sum_{n\geq 0}\left|{n \atop m}\right| x^n = \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

Another object associated with a Ferrer's diagram is the *Durfee square* which is the largest square of dots that is in the upper left corner of the Ferrer's diagram. For example for the partition $6+6+4+3+2+1+1$ we have a 3×3 Durfee square as shown in the following picture.



It should be noted that *every* partition can be decomposed into three parts. Namely a Durfee square and then two partitions, one on the right of the Durfee square and one below the Durfee square as illustrated below.



If we group partitions according to the size of the largest Durfee square and then use generating functions for partitions with at most i parts and partitions with parts at most i (which by transposition are the same generating function) we get the following identity:

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= \sum_{n\geq 0}\Bigg(\underbrace{x^{n^2}}_{\text{square}}\underbrace{\prod_{k=1}^{n}\frac{1}{1-x^i}}_{\substack{\text{left}\\ \text{partition}}}\underbrace{\prod_{k=1}^{n}\frac{1}{1-x^i}}_{\substack{\text{bottom}\\ \text{partition}}}\Bigg)\\ &= \sum_{n\geq 0}\frac{x^{n^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^n)^2}.\end{aligned}$$

There are a lot more fascinating and interesting things that we can talk about in regards to partitions. We finish our discussion with the following.

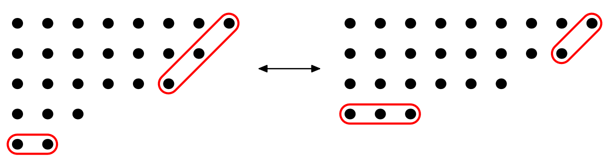
$$\begin{aligned}\prod_{n\geq 1}(1-x^i) &= 1-z-z^2+z^5+z^7-z^{12}-z^{15}+\cdots\\ &= \sum_{-\infty<j<\infty}(-1)^i z^{(3j^2+j)/2}\end{aligned}$$

This is very similar to what we encountered in the previous lecture, namely $\prod(1+x^i)$. We saw that this last expression was used to count the number of partitions with distinct parts. In the current expression something similar is going on, the difference is that previously every partition of n with distinct parts contributed 1 to the coefficient of x^n; now the contribution of a partition of n with distinct parts depends on whether the number of parts is even or odd. Namely, if the number of parts is even then the contribution is 1 and if the number of parts is odd the contribution is -1. In particular, it is not hard to see that

$$\prod_{n\geq 1}(1-x^i) = \sum_{n\geq 0}\big(E(n)-O(n)\big)x^n,$$

where $E(n)$ is the number of partitions of n into an even number of distinct parts and $O(n)$ is the number of partitions of n into an odd number of distinct parts. So to determine the product we need to determine $E(n) - O(n)$, we will see that this value is -1, 0 or 1. To do this we give a bijection between the number of partitions of n into an even number of distinct parts and the number of partitions of n into an odd number of distinct parts. This is done by taking the Ferrer's diagram of a partition into distinct parts and comparing the size of the smallest part (call it q) and the size of the largest 45° run in the upper left corner (call it r). If $q > r$ then take the points from the 45° run and make it into a new part. If $q \geq r$ then take the smallest part and put it in at the end as the 45° run.

An example of what this is doing is shown below. In the partition on the left we take the smallest part and put it at the end of the first few rows while in the partition on the right we take the small 45° run at the end of the first few rows and turn it into the smallest part.



In particular this gives an involution between partitions of n into an odd number of distinct parts and partitions of n into an even number of distinct parts. All that remains is to find the fixed points of the involution, namely those partitions into distinct parts where this operation fails. It is not too hard to see that the only way that this can fail is if we have one of the two following types of partitions.

In the one on the left we have $q = r$ but we cannot move q down to the 45° line because of the overlap. In the one on the right we have $q < r$ but if we take the points on the 45° line it will create a partition which does not have distinct parts. Counting these exceptional cases are partitions of size

$$\frac{3j^2 \pm j}{2}.$$

Putting all of this together (along with a little bit of playing with terms) explains the form of the product.

This is known as Euler's pentagonal theorem because the numbers that come up in the powers with nonzero coefficients are pentagonal numbers (which can be formed by stacking a square on top of a triangle).

The pentagonal theorem has a nice application which is useful for quickly generating the number of partitions of n.

We have

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

where the terms on the right hand side are those from the pentagonal theorem.

This follows by noting that

$$\begin{aligned} 1 &= \Big(\prod_{n\geq 0} \frac{1}{1-x^n}\Big)\Big(\prod_{n\geq 1}(1-x^n)\Big) \\ &= \Big(\sum_{n\geq 0} p(n)x^n\Big)\Big(\sum_{-\infty<j<\infty} (-1)^j x^{(3j^2+j)/2}\Big) \end{aligned}$$

Now using the rule for multiplying functions together and comparing coefficients on the left and right hand sides we see that for $n \geq 1$ that

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \cdots,$$

rearranging now gives the desired result.

We now return to generating functions. The type of generating functions that we have encountered are what are called *ordinary generating functions* which are useful in problems involving combinations. But there is another class of generating functions known as *exponential generating functions* which are useful in problems involving arrangements.

Type of generating function	form	used to count
ordinary	$\sum_{n\geq 0} a_n x^n$	combinations
exponential	$\sum_{n\geq 0} a_n \frac{x^n}{n!}$	arrangements

The term "exponential" comes from the fact that if $a_1 = a_2 = \cdots = 1$ then the resulting function is

$$\sum_{n\geq 1} \frac{x^n}{n!} = e^x.$$

A typical problem involving combinations looks at counting the number of nonnegative solutions to an equation of the form

$$n_1 + n_2 + \cdots + n_k = n$$

where we have restrictions on the n_i. In particular for each solution that we find we add 1 to the count for the number of combinations. For arrangements we start with the same basic problem, namely we must first choose what to arrange so we have to have a nonnegative solution to an equation similar to the form

$$n_1 + n_2 + \cdots + n_k = n$$

where again we have restriction on the n_i. But now after we have chosen our terms we still need to arrange (or order) them. From a previous lecture if we have n_1 objects of type 1, n_2 objects of type 2, and so on then the number of ways to arrange them is

$$\frac{n!}{n_1!n_2!\cdots n_k!},$$

so now for each solution to $n_1 + n_2 + \cdots + n_k = n$ this is what will be contributed to the count for the number of arrangements. The exponential function is perfectly set up to count the number of exponential functions.

To see this consider,

$$\Big(\sum_{\ell\in S_1}\frac{x^\ell}{\ell!}\Big)\Big(\sum_{\ell\in S_2}\frac{x^\ell}{\ell!}\Big)\cdots\Big(\sum_{\ell\in S_k}\frac{x^\ell}{\ell!}\Big)$$

where S_i is the possible values for n_i. Then a typical product will be of the form

$$\frac{x^{n_1}}{n_1!}\cdot\frac{x^{n_2}}{n_2!}\cdots\frac{x^{n_k}}{n_k!}$$

if $n_1+n_2+\cdots+n_k = n$ then the contribution to $x^n/n!$ is

$$\frac{n!}{n_1!n_2!\cdots n_k!}\frac{x^n}{n!},$$

the number of arrangements, just like we wanted!

Let us look at two related problems to compare the differences of the two types of generating functions.

Example: Find a generating function which counts the number of ways to pick n digits from 0s, 1s and 2s so that there is at least one 0 chosen and an odd number of 1s.

Solution: Using techniques from previous lectures we have that the (ordinary) generating function is

$$\begin{aligned}&(x+x^2+x^3+\cdots)(x+x^3+x^5+\cdots)(1+x+x^2+\cdots)\\&\quad=\frac{x}{1-x}\frac{x}{1-x^2}\frac{1}{1-x}=\frac{x^2}{(1-x)^3(1+x)}.\end{aligned}$$

Example: Find a generating function which counts the number of ternary sequences (sequences formed using the digits 0, 1 and 2) of length n with at least one 0 and an odd number of 1s.

Solution: This problem is related to the previous, except that after we pick out which digits we will be using we still need to arrange the digits. So we will use an exponential generating function to count these. In our case we have

$$\begin{aligned}&\big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\big)\big(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\big)\big(1+x+\frac{x^2}{2!}+\cdots\big)\\&=(e^x-1)\big(\frac{e^x-e^{-x}}{2}\big)e^x=\frac{1}{2}(e^{3x}-e^{2x}-e^x-1).\end{aligned}$$

Note that the approach is nearly identical for both problems, the only difference in the (initial) setup for the generating functions is the introduction of the factorial terms. So the same techniques and skills that we learned in setting up problems before still holds.

The only real trick is to decide when to use an exponential generating function and when not to use it. In the end it boils down (for now) to whether or not we need to account for ordering in what we choose. If we don't then go with an ordinary generating function, if we do go with an exponential generating function.

Lecture 10 – April 20

In the last lecture we saw how to set up an exponential generating function to solve an arrangement problem. As with ordinary generating functions, if we want to use exponential generating functions we need to be able to find coefficients of the functions that we set up. To do this we list some identities.

$$\begin{aligned}e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!}\\ e^{\ell x} &= \sum_{k=0}^{\infty}\ell^k\frac{x^k}{k!}\\ \frac{e^x+e^{-x}}{2} &= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\\ \frac{e^x-e^{-x}}{2} &= x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\\ \sum_{k\ge n}\frac{x^k}{k!} &= e^x-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots-\frac{x^{n-1}}{(n-1)!}\end{aligned}$$

Example: Find the number of ternary sequences (sequences formed using the digits 0, 1 and 2) of length n with at least one 0 and an odd number of 1s.

Solution: From last time we know that the exponential generating function $g(x)$ which counts the number of ternary sequences is

$$\begin{aligned}g(x) &= \frac{1}{2}(e^{3x}-e^{2x}-e^x-1)\\ &= \frac{1}{2}\Big(\sum_{n\ge0}3^n\frac{x^n}{n!}-\sum_{n\ge0}2^n\frac{x^n}{n!}-\sum_{n\ge0}\frac{x^n}{n!}-1\Big)\\ &= \sum_{n\ge1}\frac{1}{2}\big(3^n-2^n-1\big)\frac{x^n}{n!}.\end{aligned}$$

So reading off the coefficient of $x^n/n!$ we have that the number of such solutions is $(3^n-2^n-1)/2$ for $n\ge1$ and 0 for $n=0$.

Example: Find the exponential generating function where the nth coefficient counts the number of n letter words which can be formed using letters from the word "BOOKKEEPER".

Solution: We will use an exponential generating function since we are interested in counting the number of words and the arrangement of letters in words gives different words. Since order is important we will go with an exponential generating function. We group by letters. There are 3 Es, 2 Ks and Os and 1 B, P and R. So putting this together the desired exponential generating function is

$$\begin{aligned} f(x) &= \big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\big)\big(1+x+\frac{x^2}{2!}\big)^2\big(1+x\big)^3 \\ &= 1+6x+33\frac{x^2}{2!}+166\frac{x^3}{3!}+758\frac{x^4}{4!}+3100\frac{x^5}{5!} \\ &\quad +11130\frac{x^6}{6!}+34020\frac{x^7}{7!}+84000\frac{x^8}{8!} \\ &\quad +151200\frac{x^9}{9!}+151200\frac{x^{10}}{10!}. \end{aligned}$$

(The last step is done by multiplying the polynomial out, letting the computer do all of the heavy lifting. So we can read off the coefficients and see, for example, that there are 34020 words of length 7 that can be formed using the letters from BOOKKEEPER.)

Example: How many ways are there to place n distinct people into three different rooms with at least one person in each room?

Solution: This does not look like an arrangement problem, so we might not automatically think of using exponential generating functions. However, we previously saw that the number of ways of arranging n_1 objects of type 1, n_2 objects of type 2, …, and n_k objects of type k is the same as the number of ways to distribute n distinct objects into k bins so that bin 1 gets n_1 objects, bin 2 gets n_2 objects, …, and bin k gets n_k objects.

So this problem is perfect for exponential generating functions. In particular, since there are three room we can set this up as $n_1 + n_2 + n_3 = n$ where n_i are the number people in room i. Since each room must have at least one person we have that $n_i \geq 1$. So the exponential generating function is

$$\begin{aligned} h(x) &= \big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\big)^3 \\ &= \big(e^x-1\big)^3 \\ &= e^{3x}-3e^{2x}+3e^x-1 \\ &= \sum_{n\geq 0} 3^n\frac{x^n}{n!}-3\sum_{n\geq 0}2^n\frac{x^n}{n!}+3\sum_{n\geq 0}\frac{x^n}{n!}-1 \\ &= \sum_{n\geq 1}\big(3^n-3{\cdot}2^n+3\big)x^n. \end{aligned}$$

Reading off the coefficients we see that the answer to our question with n people is $3^n - 2{\cdot}2^n + 3$. In a later lecture we will see an alternative way to derive this expression using the principle of inclusion-exclusion.