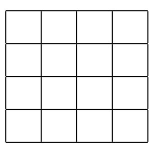
Lecture 1 – March 30

We start our investigation into combinatorics by focusing on *enumerative* combinatorics. This topic deals with the problem of answering "how many?" We will start with a set S, which is a collection of objects called elements, and then we will determine the number of elements in our set denoted $|S|$. (More information about the basics of set theory are found in the appendix of *Applied Combinatorics*.)

We start with a few basic rules for counting.

> Addition Rule
> If the set we are counting can be broken into *disjoint* pieces, then the size of the set it the *sum* of the size of the pieces.
> $$S = \bigcup_{i=1}^{m} S_i,\ S_i \cap S_j = \emptyset \text{ if } i \neq j \ \Rightarrow\ |S| = \sum_{i=1}^{m} |S_i|.$$

Example: A square with side length 4 is divided into 16 equal sqaures as shown below. What is the total number of squares in the picture?



Solution: We break the squares up according to size. There are sixteen 1×1 squares, nine $2\times w$ squares, four 3×3 squares and one 4×4 square. So by the addition rule there is a total of $16+9+4+1=30$ squares. (Note: here we must be careful about what is meant by disjoint. Geometrically as squares the squares are not all disjoint one from another, but we are not concerned with the geometry of the problem so when we talk about disjoint we mean not the same square. So in our case when we broke up the squares into these sets by size theses sets are disjoint one from another.)

> Multiplication Rule
> Suppose we can describe the elements in our set using a procedure with m steps, where at the ith step we have r_i choices available. (Where the *number* of choices is independent of our previous choices.) Then the size of the set is $r_1 r_2 \cdots r_m$.

It is important to realize that what we are counting using the multiplication rule is the number of choices that can be made. In order to guarantee that we get the right count it is important that (1) every element in our set is realizable by at least one set of choices and (2) no two sets of choices gives us the same element. An example of what can go wrong will be given in a later lecture.

Another important thing to note is that the number of choices is independent on the previous choices but not necessarily the choice that we have to make. This is illustrated in the next example.

Example: How many standard California license plates are possible? A standard license plate has the form $NLLLNNN$ where N is a number and L is a letter. How many are possible if there is no repetition of numbers or letters?

Solution: We can form the license plate one character at a time. So we will use the multiplication rule where the decision at the ith stage is what character goes in the ith slot. Since there are 10 possible numbers and 26 possible letters we have that the number of license plates is

$$10\cdot26\cdot26\cdot26\cdot10\cdot10\cdot10 = 175,760,000.$$

When we add the restriction that that there is no repetition of characters we again use the same principle as before. Now the difference is that when we fill in the second number we only have nine options since we have already used one number, when we fill in the third we only have eight options and when we fill in the fourth we only have seven. Note that here we do not know which nine, eight or seven choices are available because that depends on previous choices, but the number of choices does not! So the number when there is no repetition of numbers is

$$10\cdot26\cdot25\cdot24\cdot9\cdot8\cdot7 = 78,624,000.$$

> Bijection Rule
> If the elements in S can be paired in a one-to-one (i.e., bijective) fashion, then $|S| = |T|$.

We have already used the bijection rule when we did the multiplication rule, since what we are doing is counting the number of choices which are paired one-to-one with the elements. One interesting thing is that it is possible to show that two sets have equal size by giving a bijection between the sets without ever knowing the size of the sets themselves.

Example: How many subsets of $[n] = \{1,2,\ldots,n\}$ are there?

Solution: We can pair a subset of $[n]$ with a binary word of length n (a binary word is a word with the individual letters coming from 0 and 1). The way this is done is by taking a subset A and letting the ith letter of the binary word be 1 if i is in A and 0 if i is not in A. For example for $\{1,2,3\}$ we have

$$\begin{array}{rr} \emptyset \leftrightarrow 000 & \{3\} \leftrightarrow 001 \\ \{1\} \leftrightarrow 100 & \{1,3\} \leftrightarrow 101 \\ \{2\} \leftrightarrow 010 & \{2,3\} \leftrightarrow 011 \\ \{1,2\} \leftrightarrow 110 & \{1,2,3\} \leftrightarrow 111 \end{array}$$

Since there are 2^n binary words it follows by the bijection rule that the number of subsets is also 2^n.

> Rule of Counting in Two Ways
> If we count S in two different ways the results must be equal.

This is the basic idea behind many combinatorial proofs. Namely we find an appropriate set and count it in two different ways. By then setting them equal we get some nontrivial relationships.

Example: Count the number of $*$'s in the diagram which has n rows and $n+1$ columns (below we show the case $n = 4$) in two different ways.

$$\begin{matrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}$$

Solution: For our first method we count along rows. Each row has $(n+1)$ of the $*$'s and there are n rows so in total we have $n(n+1)$ total $*$'s. For our second method we count using the diagonals. In particular we have that there are

$$1+2+3+\cdots+n+n+\cdots+3+2+1 = 2\sum_{i=1}^{n} i.$$

Equating these two ways of counting we have

$$n(n+1) = 2\sum_{i=1}^{n} i \quad \text{or} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

> Probability is counting
> Probability is a measurement of how likely an outcome is to occur. If we are dealing with finitely many possible outcomes and each outcome is equally likely (very common assumptions!) then the probability that a certain outcome occurs is the ratio of the outcomes with the desired result divided by the number of all possible outcomes.

Lecture 2 – April 1

Example: A domino is a tile of size 1×2 divided into two squares. On each square are pips (or dots), the number of pips possible are$0, 1, 2, 3, 4, 5, 6$. There are 28 possible domino pieces, i.e., [0|0], [0|1], [0|2], ..., [6|6]. Two domino pieces fit if we can arrange them so the two adjacent squares have the same number of pips. So for example [1|3] [3|6] fit but [1|3] [2|6] do not fit. What is the probability that two dominos chosen at random will fit?

Solution: First let us count the total number of ways to pick two dominos. We can think of choosing one domino and then a second one. This can be done in $28\cdot27 = 756$ ways. But we have overcounted, this is because the order that we pick domino pieces should not matter. So for example right now we distinguish between the choice [1|3] [5|6] and the choice [5|6] [1|3]. In essence we have counted every pair twice. To correct for this we divide by 2 so that the number of ways to pick two domino pieces is $756/2 = 378$.

Next let us count the number of pairs for which the dominos fit. We again think of picking a domino and then seeing how many domino pieces fit with it. There are two cases, if we first pick a domino of the form [x|x] (of which there are seven of this form) then dominos which fit with it are of the form [x|y] with $y \neq x$, so there are six matches. So in this case we got $6\cdot7 = 42$ pairs. In the second case we have a domino of the form [x|y] where $x \neq y$, there are 21 such dominos and there are twelve dominos that can fit with this type of domino, namely, [x|z] for $z \neq x$ and [y|z] for $z \neq y$. So in this case we got $21\cdot12 = 252$ pairs. Combining we have $42 + 252 = 294$ pairs that match, but as before we have counted every pair twice and so there are 147 pairs of dominos that fit.

Combining we have that the probability that two dominos chosen at random will fit is $147/378 = 7/18$.

Given a set with n objects, a *permutation* is an ordered arrangement of all n of the objects. An *r-permutation* is an ordered arrangement of r of the n objects. An *r-combination* is an unordered selection of r of the n objects.

We now count how many of each type of the following there are. First off for a permutation by the multiplication rule we choose which object will be first (we have n choices), then we choose which object will be second (we now have $n-1$ choices), and continue so on to the end. So there are

$$n(n-1)(n-2)\cdots3\cdot2\cdot1 = n! \quad \text{permutations.}$$

The notation $n!$, read "n factorial", arises frequently in combinatorics and it is important to get a good handle on using it. For example, it is useful to be able to rewrite factorials, i.e., $(2n+2)! = (2n+2)(2n+1)(2n)!$. For convenience we will define $0! = 1$ (this helps to simplify a lot of notation).

Another useful fact for factorials is Stirling's approximation which states that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}.$$

This is useful when trying to get a handle on the size of an expression that involves factorial. For instance, one object which has been extensively studied are the middle binomial coefficients (see below) $\binom{2n}{n}$, which by Sterling's approximation we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}} = \frac{1}{\sqrt{\pi n}}4^n.$$

The method to count r-permutations is the same as that of counting permutations. We use the multiplication principle but instead of going through n stages we stop at the rth stage. So we have that there are

$$\begin{aligned} &n(n-1)(n-2)\cdots\big(n-(r-1)\big) \\ &= \frac{n(n-1)(n-2)\cdots\big(n-(r-1)\big)(n-r)\cdots 2\cdot 1}{(n-r)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} = P(n,r). \end{aligned}$$

This is also sometimes written as the falling factorial $(n)_k$ or $n^{\underline{k}}$.

To count r-combinations we can count r-permutations in a different way. Suppose that we let $C(n,r)$ be the number of ways to pick r unordered objects, then we have

$$P(n,r) = C(n,r)r!.$$

Namely, to have an ordering of r elements we first pick the r elements and then we order them. Rearranging we have

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

The term $\binom{n}{r}$, read "n choose r", is also known as a binomial coefficient. This will be discussed in a future lecture.

Example: How many ways can n rooks be placed on $n \times n$ chessboard so no two threaten each other? How many ways can k rooks be placed on $n \times n$ chessboard if $k \le n$?

Solution: A rook can move along any row and column. If there are n rooks and at most one rook can go in a column then there will be a single rook in each column. We place rooks one column at a time. In the first column we have n choices, in the second column we cannot put it in the same row as we used for the first rook so there are $n-1$ choices, for the third column we have $n-2$ choices and so on. In total there are

$$n(n-1)(n-2)\cdots 2\cdot 1 = n! \quad \text{placements.}$$

If we have k rooks then the approach is similar. First we choose the columns that the rooks will go into. This can be done in $C(n,k)$ ways. Once we have the columns we place rooks in one column at a time. In particular the way that we can distribute the rooks in the k columns is

$$n(n-1)\cdots\big(n-(k-1)\big) = P(n,k).$$

Putting it together the number of ways to place the k rooks is $C(n,k)P(n,k)$.

Problems of placing rooks on chessboards can lead to some interesting (and highly non-trivial) combinatorics! These types of problems can arise in algebraic combinatorics which unfortunately we will not have time to address this quarter.

Example: Given eight points which line in the plane, no three on a line, how many lines do these points determine?

Solution: To determine a line we need two points. So this reduces to the number of ways to pick two of the eight patterns which can be done in $\binom{8}{2}$ ways. Note that it is important that no three are on a line to guarantee that we do not overcount!

Lecture 3 – April 3

Example: How many ways are there to choose k elements from $[n] = \{1, 2, \ldots, n\}$ so that no *two* are consecutive?

Solution: If we did not have the restriction that elements couldn't be consecutive then we could do this in $\binom{n}{k}$ ways. The problem is on how to guarantee that the elements are not consecutive. Suppose that our elements we choose are

$$a_1 < a_2 < a_3 < \cdots < a_k$$

where no two are consecutive. Using the fact that no two are consecutive we have that

$$a_1 < a_2 - 1 < a_3 - 2 < \cdots < a_k - (k-1).$$

These are k ditinct elements from $[n-(k-1)] = \{1, 2, \ldots, n-(k-1)\}$. On the other hand given k distinct elements $b_1 < b_2 < \cdots < b_k$ from $[n-(k-1)]$ we can find k elements from $[n]$ for which no two are consecutive, namely the elements

$$b_1 < b_2 + 1 < b_3 + 2 < \cdots < b_k + (k-1).$$

So the number of ways there are to choose k elements from $[n] = \{1, 2, \ldots, n\}$ so that no *two* are consecutive is the same as the number of ways there are to choose k elements from $[n-(k-1)]$, which is $\binom{n-(k-1)}{k}$.

When using the multiplication principle it is important to make sure that different sets of choices do not lead to the same outcome. This can sometimes be subtle to detect so great care should be taken!

Example: From among seven boys and four girls how many ways are there to choose a six member volleyball team with at least two girls?

Solution: One (seemingly) natural approach is to first pick two girls and then pick the remainder of the team. This way we guarantee that we meet the condition. The number of ways to pick the two girls is $\binom{4}{2}$, there are now nine people left and we have to choose 4 of then and this could be done in $\binom{9}{4}$ ways, for a grand total of $\binom{4}{2}\binom{9}{4} = 756$ different possible teams.

But wait! Suppose the girls are A, B, C, D and the boys $1, 2, 3, 4, 5, 6, 7$ then we could have first chosen AB and then $C246$ to form the team $ABC246$, but we could also have first chosen BC and then $A246$ to form the team $ABC246$. So different choices gave us the same team. (We have committed the cardinal sin of overcounting.)

To correct this we have two options, one is to figure out how much we have overcounted and then subtract off the overcount. The other is to go back and count in a different way, one such different method is to split up the count into smaller cases where we won't overcount. For example in this problem let us break it up by the number of girls on the team. If there are two girls we pick two girls and four boys to form the team which can be done in $\binom{4}{2}\binom{7}{4}$, if there are three girls we pick three girls and three boys to form the team which can be done in $\binom{4}{3}\binom{7}{3}$, if there are four girls we pick four girls and two boys to form the team which can be done in $\binom{4}{4}\binom{7}{2}$ ways, so altogether there are

$$\binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{3} + \binom{4}{4}\binom{7}{2} = 371 \quad \text{teams.}$$

An arrangement of n distinct objects is $n!$. But what happens if the objects are not distinct? For example, suppose we are looking at anagrams, rearrangements of letters of a word. For instance "STEVE BUTLER" can be arranged to "RUE VEST BELT" but it could also be arranged to gibberish "VTSBEEERTLU". So the number of ways there are to rearrange the letters of a given word (allowing for gibberish answers) is found by counting the arrangements of objects with repetition.

Example: How many ways are there to rearrange the letters of the word "BANANA"?

Solution: We present two methods. The first method is to group the letters together by type. We have one B, three As and two Ns. We now start with six empty slots which we will fill in with our letters in order. First we put in the B, there are six slots and we must choose one of them so this can be done in $\binom{6}{1}$ ways. We now have five slots left for us to choose for the position of the three As which can be done in $\binom{5}{3}$ ways. Finally we have two slots left for us to choose for the position of the two Ns which can be done in $\binom{2}{2}$ ways. Giving us a total of

$$\binom{6}{1}\binom{5}{3}\binom{2}{2} = 60 \quad \text{rearrangements.}$$

For our second method we first make the letters distinct. This is done by labeling, so we have the letters $B, A_1, A_2, A_3, N_1, N_2$. These letters can be arranged in 6! ways and finally we remove the labaling. The problem is that we have now overcounted. For instance for the end result of $ABANNA$ this would show up twelve ways, as

$$\begin{array}{lll} A_1BA_2N_1N_2A_3 & A_1BA_3N_1N_2A_2 & A_2BA_1N_1N_2A_3 \\ A_2BA_3N_1N_2A_1 & A_3BA_1N_1N_2A_2 & A_3BA_2N_1N_2A_1 \\ A_1BA_2N_2N_1A_3 & A_1BA_3N_2N_1A_2 & A_2BA_1N_2N_1A_3 \\ A_2BA_3N_2N_1A_1 & A_3BA_1N_2N_1A_2 & A_3BA_2N_2N_1A_1 \end{array}$$

Namely, we have 3! ways to label the As and 2! ways to label the Ns and 1! ways to label the B. This happens with each arrangement so in total we have

$$\frac{6!}{1!3!2!} = 60 \quad \text{rearrangements.}$$

These two approaches generalize. The first idea is to choose the slots for the first type of objects, then choose the slots for the second type of objects and so on. The second idea is to label the elements and permute and then divide out by the overcounting.

Given n_1 objects of type 1, n_2 objects of type 2, ..., n_k objects of type k and $n = n_1 + n_2 + \cdots + n_k$ then the number of ways to arrange these objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

The term $\binom{n}{n_1,n_2,\ldots,n_k}$ is also known as a multinomial coefficient, we will discuss these more in a later lecture.

Example: How many ways are there to rearrange the letters of "ROKOKO" so there in no "OOO"?

Solution: The number of ways to rearrange the letters of "ROKOKO" is the same as the number of ways to rearrange the letters of "BANANA", so there are 60 in all. But of course some of these have "OOO" but we can use one of the most useful techniques in counting:

Sometimes it is easier to count the complement.

So we now count the number of arrangements with "OOO". This can be done by considering the arrangements of the letters R,K,K,OOO. This can be done in $4!/2! = 12$ ways. So the total number of rearrangements without "OOO" is $60 - 12 = 48$.

A related problem is to distribute n identical objects among k different boxes (or people, or days, or so on). One way to do this is to use $*$'s (stars) to denote the identical objects and lay them down in a row. We then draw in $k-1$ dividing lines (bars) which divides the n objects into k groups, the first group goes in the first box, the second group goes in the second box and so on.

For example suppose we are distributing $n = 20$ pennies among $k = 6$ children. Then using bars and stars we represent giving four pennies to the first child, two to the second, six to the third, none to the fourth, five to the fifth and three to the sixth by

$$****|**|******||*****|***.$$

In this case we have a total of 25 bars and stars and once we know where the bars (or stars) go then we know where the stars (or bars) go. So the number of ways to do the distribution is the number of ways to pick the bars (or stars) which is $\binom{25}{5} = \binom{25}{20}$.

The number of ways to distribute n identical objects into k distinguished boxes is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Example: How many ways are there to distribute twenty pennies among six children? What if each child must get at least one penny?

Solution: We already have answered the first part, this can be done in $\binom{25}{5} = 53,130$ ways. For the second part we distribute pennies in two rounds. In the first round we give one penny to each child, thus satisfying the condition that each child gets a penny. In the second round we distribute the remaining fourteen pennies among the six children arbitrarily which can be done in $\binom{19}{5} = 11,628$ ways.

Lecture 4 – April 6

Example: How many ways are there to order six 0s, five 1s and four 2s so that the first 0 occurs before the first 1 which in turn occurs before the first 2?

Solution: We can build up the words one group of letters at a time. To simplify the process we first put down the 2s, then the 1s and finally the 0s. First putting down the 2s we have

2222

Next we put down the 1s which can go before and after the 2s, i.e.,

␣2␣2␣2␣2␣.

To satisfy the constraint we must have one of the 1s go into the first slot, and the remaining $n = 4$ 1s are distributed among the $k = 5$ slots which can be done in $\binom{8}{4}$ ways. We now need to decide how to put in the 0s, these again can go before and after the letters already placed, i.e.,

␣1␣*␣*␣*␣*␣*␣*␣*␣*␣.

To satisfy the constraint we must have one of the 0s go into the first slot, and the remaining $n = 5$ 0s are distributed among the $k = 10$ slots which can be done in $\binom{14}{9}$ ways. Combining this gives us

$$\binom{8}{4}\binom{14}{9} = 140,140 \quad \text{ways.}$$

Many problems into combinatorics relate to the problem of placing n balls into k bins. The number of ways to do this is dependent on our assumptions.

- *Identical balls and distinct bins:* This is what was discussed in the previous lecture, so it can be done in
$$\binom{n+(k-1)}{k-1} \quad \text{ways.}$$
- *Distinct balls into distinct bins:* Imagine we place the balls one at a time. The first ball can go into any of k bins, the second ball can go into any of k bins, ..., the nth ball can go into any of k bins. So by the multiplication rule the number of ways that this can be done is
$$\underbrace{k \cdot k \cdot \cdots \cdot k}_{n \text{ times}} = k^n.$$
- *Distinct balls into distinct bins, with number of balls in each bin specified:* That is we want to place the balls so that n_1 balls go into the first bin, n_2 balls go into the second bin, ..., n_k balls go into the kth bin, where $n = n_1 + \cdots + n_k$. This can be done by choosing n_1 balls for the first bin, then choose n_2 of the remaining balls for the second bin, n_3 of the now remaining balls for the third bin, and so on. The number of ways to do this is
$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k},$$
which from the last lecture is the same as
$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

- *Identical balls into identical bins:* This looks at how to break up the n balls into groups where order is unimportant. This leads to the field of *partitions* (an area of combinatorics with really beautiful proofs and really hard problems), we will be looking at these in a later lecture.

- *Distinct balls into identical bins:* This looks at how to break up $[n] = \{1, 2, \ldots, n\}$ into small subsets. This can be counted using Bell numbers and Stirling numbers which we will look at (briefly) in a later lecture.

The number of ways of distributing n identical objects into k bins has another interpretations, namely the number of nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

To see this we count the balls, let x_i be the number of balls in the ith bin, then the sum of the x_i is the total number of balls which is n. Conversely, given a solution of x_i we can make a distribution into the bins by putting x_i balls into the ith bin for each i.

Example: Count the number of integer solutions for each of the following.

1. $x_1 + x_2 + x_3 + x_4 = 10$ with $x_i \geq 0$.
2. $x_1 + x_2 + x_3 + x_4 = 10$ with $x_i \geq 1$.
3. $x_1 + x_2 + x_3 + 3x_4 = 10$ with $x_i \geq 0$.
4. $x_1 + x_2 + x_3 + x_4 \leq 10$ with $x_i \geq 0$.

Solution: We have the following.

1. This is the straightforward case of $n = 10$ and $k = 4$ variables so this can be done in $\binom{13}{3} = 286$ ways.
2. We satisfy the constraint $x_i \geq 1$ by first distributing 1 to each x_i, we now distribute the remaining $n = 6$ among the $k = 4$ variables so this can be done in $\binom{9}{3} = 84$ ways.
3. The difficult part about this is the x_4 term, so we break into cases depending on the value of x_4. If $x_4 = 0$ we distribute $n = 10$ among $k = 3$ which can be done in $\binom{12}{2}$ ways. If $x_4 = 1$ we distribute $n = 7$ among $k = 3$ which can be done in $\binom{9}{2}$ ways. If $x_4 = 2$ we distribute $n = 4$ among $k = 3$ which can be done in $\binom{6}{2}$ ways. Finally, if $x_4 = 3$ we distribute $n = 1$ among $k = 3$ which can be done in $\binom{3}{2}$ ways. So by the addition rule the total number of solutions is
$$\binom{12}{2} + \binom{9}{2} + \binom{6}{2} + \binom{3}{2} = 120 \text{ ways.}$$
4. We could break this up as we did the previous case and get the sum of several terms. However we can avoid all of this by considering the auxiliary problem
$$x_1 + x_2 + x_3 + x_4 + x_5 = 10 \text{ with } x_i \geq 0.$$
Here the x_5 represents what we are "short" by in our sum to 10. In particular each solution to this problem is a solution to our original problem, giving us $\binom{14}{4} = 1,001$ ways.

Example: How many ways are there to rearrange the letters of the word "`MATHEMATICASTER`" so there are no consecutive vowels?

Solution: We break this into three steps. First the constraint gives us relationships between the consonants and the vowels so we will see how many vowel-consonant patterns there are. There are six vowels and nine consonants. If we put down the six vowels then we need to decide how to distribute the consonants among the vowels, i.e.,

`␣V␣V␣V␣V␣V␣V␣`.

To satisfy our constraint we have to put five of the consonants in slots between the vowels, we can distribute the remaining $n = 4$ consonants among the $k = 7$ slots which can be done in $\binom{10}{6}$ ways. Given the pattern we now arrange the vowels and consonants which can be done respectively in

$$\frac{6!}{3!2!1!} \text{ and } \frac{9!}{3!2!1!1!1!1!} \text{ ways.}$$

Combining everything we have

$$\binom{10}{6} \cdot \frac{6!}{3!2!1!} \cdot \frac{9!}{3!2!1!1!1!1!} = 381,024,000 \text{ ways.}$$

The *binomial theorem* looks at expressions of the form $(x + y)^n$. The term binomial refers to the two nomials, or variables, x and y. For the first few cases we have:

$$\begin{aligned} (x+y)^0 &= 1 \\ (x+y)^1 &= x+y \\ (x+y)^2 &= x^2+2xy+y^2 \\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4 \end{aligned}$$

In general we have

$$\begin{aligned} (x+y)^n &= \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ terms}} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \end{aligned}$$

The idea behind the term $\binom{n}{k}$ is that the way we multiply out $(x+y)^n$ is to choose one term from each of the n copies of $(x+y)$, to get $x^k y^{n-k}$ we have to choose the x value for exactly k of the n possible places which can be done in $\binom{n}{k}$. Because of its connection to the binomial theorem the terms $\binom{n}{k}$ are also known as *binomial coefficients*.

There is a more general version of the binomial attributed to Isaac Newton. First we define for any number r (even imaginary!) and integer $k \geq 1$

$$\binom{r}{k} = \frac{r(r-1)\cdots\big(r-(k-1\big)}{k!}.$$

Further we let $\binom{r}{0} = 1$. Then for $|x| < 1$

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

In the case that r is an integer then the term $\binom{r}{k}$ becomes 0 for k sufficiently large so this gives us back the form of the binomial theorem given above.

Lecture 5 – April 8

As an application of the binomial theorem we have the following.

> Let $e = 2.718281828\ldots$ and $0 \leq k \leq n$, then
> $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Proof: For the first inequality we have

$$\begin{aligned}\binom{n}{k} &= \frac{n}{k}\frac{(n-1)}{(k-1)}\frac{(n-2)}{(k-2)}\cdots\frac{n-(k-1)}{k-(k-1)} \\ &\geq \frac{n}{k}\frac{n}{k}\frac{n}{k}\cdots\frac{n}{k} \\ &= \left(\frac{n}{k}\right)^k,\end{aligned}$$

where we used the fact that $(n-a)/(k-a) \geq n/k$ for $0 \leq a \leq k-1$. For the other inequality we first note that from calculus we have that $e^x \geq 1+x$ for all x, In particular for $x \geq 0$ we have

$$\begin{aligned}e^{nx} &\geq (1+x)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}x^i \\ &\geq \binom{n}{k}x^k.\end{aligned}$$

Now choose $x = k/n$ and simplify to get the result. □

Today we will look at using the binomial theorem and more generally at the binomial coefficients. Our starting point will be looking at the coefficients in the binomial theorem, i.e., $\binom{n}{k}$. These can be arranged in a triangle known as Pascal's triangle as follows:

$$\begin{array}{c}\binom{0}{0}\\ \binom{1}{0}\ \binom{1}{1}\\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}\\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}\\ \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}\\ \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}\\ \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}\end{array}$$

Inserting the values of these numbers we have

$$\begin{array}{c}1\\ 1\quad 1\\ 1\quad 2\quad 1\\ 1\quad 3\quad 3\quad 1\\ 1\quad 4\quad 6\quad 4\quad 1\\ 1\quad 5\quad 10\quad 10\quad 5\quad 1\\ 1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\end{array}$$

These numbers occur frequently in combinatorics so it is good to have the first few rows memorized and know some of the basic properties of numbers in this triangle. We will now start to describe some of the various properties of the numbers in this triangle.

One thing which is apparent is that the rows are symmetric. In terms of the binomial coefficients this says the following.

> $$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic proof: Using the definition of $\binom{n}{k}$ from a previous lecture we have

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-k)!\big(n-(n-k)\big)!} = \binom{n}{n-k}.\end{aligned}$$ □

Combinatorial proof: By definition $\binom{n}{k}$ is the number of ways to choose k elements from a set of n elements. This is the same as deciding which $n-k$ numbers will not be chosen which can be done in $\binom{n}{n-k}$ ways. □

We have now given two proofs. Certainly one proof is sufficient to show that it is true. Nevertheless it is useful to have many different proofs of the same fact (for instance there are hundreds of proofs for the Pythagorean Theorem and new ones are constantly being discvoered), since they can give us different ideas of how to use these tools in other problems. By "combinatorial proof" we mean a proof wherein we count some object in two different ways (i.e., using the *Rule of Counting in Two Ways* from the first lecture).

Another pattern that seems to be happening is that the terms in the rows first increase until the halfway point and then they decrease. This behavior is called unimodal and we have the following.

For n fixed, $\binom{n}{k}$ is unimodal.

Proof: To show this we have to show that it first will increase and then decrease. This can be done by looking at the ratio of consecutive terms. In particular we have

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} \begin{cases} \geq 1 & \text{then } \binom{n}{k} \geq \binom{n}{k-1}, \\ \leq 1 & \text{then } \binom{n}{k} \leq \binom{n}{k-1}. \end{cases}$$

Substituting in the definition for the binomial coefficient we have

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} = \frac{n-k+1}{k}.$$

Solving we see that $\binom{n}{k}/\binom{n}{k-1} \geq 1$ when $k \leq (n+1)/2$. In particular we see that for the first half of a row on Pascal's triangle that the binomial coefficients will increase and for the second half of the row they will decrease. □

Actually more can be said about the size of the binomial coefficients. Namely, it can be shown that they form the (infamous) bell shaped curve.

One very important pattern is how the coefficients of one row relate to the coefficients of the previous row. This gives us the most important identity for binomial coefficients.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

In other words this says to find the term $\binom{n}{k}$ in Pascal's triangle you look at the two numbers above and add them. Using this one can quickly generate the first few rows (so instead of memorizing the rows you can also memorize the first three or four and then memorize how to fill in the rest). This also can be used to give an inductive proof for the binomial theorem.

Algebraic proof: Plugging in the definitions for binomial coefficients we have

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}. \quad \square\end{aligned}$$

Combinatorial proof: We have that $\binom{n}{k}$ is the number of ways to select k elements from $\{1, 2, \ldots, n\}$. Using the addition rule this is the number of ways to select k elements from $\{1, 2, \ldots, n\}$ with n being one of the chosen elements added to the number of ways to select k elements from $\{1, 2, \ldots, n\}$ with n not being one of the chosen elements. In the first case there are $\binom{n-1}{k-1}$ ways to choose the remaining $k-1$ elements other than n and in the second case there are $\binom{n-1}{k}$ ways to choose k elements other than n. □

If we sum the values in the first few rows of Pascal's triangle we see that we get $1, 2, 4, 8, 16, 32, 64$. This is a nice pattern and it holds in general.

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Algebraic proof: Putting $x = y = 1$ into the binomial theorem we have

$$2^n = (1+1)^n = \sum_{k=0}^{n} 1^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}. \quad \square$$

Combinatorial proof: In a previous lecture we saw that the number of subsets of $\{1, 2, \ldots, n\}$ is 2^n. On the other hand the number of subsets with k elements is $\binom{n}{k}$. Combining these two ideas we have that the number of subsets is the number of subsets of size 0 added to the number of subsets of size 1 added to the number of subsets of size 2 ... added to the number of subsets of size n. □

We can also use the binomial theorem in more subtle ways.

$$\sum_{k=0}^{n} k\binom{n}{k} = 1\binom{n}{1} + 2\binom{n}{2} \cdots + \binom{n}{n} = n2^{n-1}.$$

Algebraic proof: Thinking of the binomial theorem in terms of a function of x we have

$$\begin{aligned} n(x+1)^{n-1} &= \frac{d}{dx}\big((x+1)^n\big) \\ &= \frac{d}{dx}\left(\sum_{k=0}^{n} x^k\binom{n}{k}\right) = \sum_{k=0}^{n} kx^{k-1}\binom{n}{k}. \end{aligned}$$

Substituting $x = 1$ on the left and right sides gives us the result. □

Of course this also has a combinatorial interpretation.

Combinatorial proof: Examining the term $k\binom{n}{k}$ this counts the number of ways to pick a k-element subset and then pick one of these elements. We can interpret this as the number of ways of forming a committee of size k with a chairperson, i.e., first we pick the k people who will be on the committee and then we select one of the chosen people to be the chair. So the left hand side, which sums over all possible committee sizes, counts all possible ways to form a committee with a chairperson.

We can also count this by first selecting the chairperson (which can be done in n ways) and then choosing the rest of the committee. Since there are $n-1$ people available to serve on the committee with the chair we can fill in the rest of the committee in 2^{n-1} ways. □

Counting committees arrangements gives some simple arguments to some binomial identities. As another example of this type of committee forming argument consider the following.

$$\boxed{\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.}$$

Combinatorial proof: Let us count the number of ways to form a committee of size r from a group of m women and n men. If we ignore the gender balance, since there are $m+n$ people then this can be done in $\binom{m+n}{r}$ ways. On the other hand the number of ways to form a committee with exactly k women is to first select the women in $\binom{m}{k}$ ways and then fill up the remaining seats with the men in $\binom{n}{r-k}$ ways. So $\binom{m}{k}\binom{n}{r-k}$ is the number of ways to form our committee of total size r with exactly k women. Since the possible values for k are $0, 1, 2, \ldots, r$ the result now follows by the rule of addition. □

There is also an algebraic proof. We do not give all the details but only provide a brief sketch: The right hand side is the coefficient of x^r in the expansion of $(1+x)^{m+n}$, the left hand side is the coefficient of x^r in the expansion of $(1+x)^m(1+x)^n$. Since $(1+x)^{m+n} = (1+x)^m(1+x)^n$ the coefficients must be equal giving the result.

As a special case of the previous result we have the following.

$$\boxed{\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.}$$

In other words if we add up the square of the numbers in a row of Pascal's triangle the result is another binomial coefficient. To see how this follows from what we just proved we note that using the symmetry of the binomial coefficients

$$\begin{aligned} &\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \\ &\quad= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n}\binom{n}{n} \\ &\quad= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} \\ &\quad= \binom{2n}{n}. \end{aligned}$$

The last step follows from noting that this is the previous result with $m = r = n$.

The following is a useful fact for rewriting product of binomial coefficients.

$$\boxed{\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.}$$

Algebraic proof: Plugging in the definitions for the binomial coefficients we have

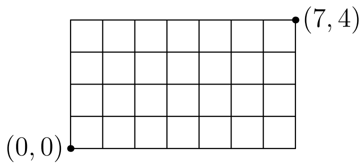
$$\begin{aligned} \binom{n}{m}\binom{n-m}{k-m} &= \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!(n-k)!} \\ &= \frac{n!}{(n-k)!m!(k-m)!} = \frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} \\ &= \binom{n}{k}\binom{k}{m}. \end{aligned}$$ □

Combinatorial proof: Examining the left hand side we first pick k out of n elements and then we pick m out of k elements. We can interpret this as the number of sets A and B so that $A \subseteq B \subseteq X$ with $|A| = m$, $|B| = k$ and $|X| = n$. That is, given an n element set X we first pick B as a k element subset of X and then pick A as an m element subset of B.

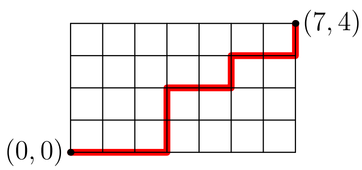
We could also count this in a different way. Namely, we first pick the set A (which can be done in $\binom{n}{m}$ ways and then pick B so that $A \subseteq B \subseteq X$. To do the second step we need to "fill out" the set B by adding an additional $k-m$ elements to the k already chosen for A, since there are $n-m$ elements not in A to choose from we can do this in $\binom{n-m}{k-m}$ ways. □

We have already seen how to use committees to give identities about binomial coefficients. Another useful object is studying walks on a square lattice.

Example: On the square lattice shown below how many different walks are there from $(0,0)$ to $(7,4)$ which consists of steps to the right by one unit or up one unit?

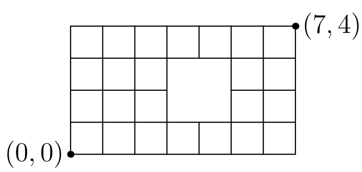


Solution: Looking at the walks we see that we will need to take 7 steps to the right and 4 steps up, for a total of 11 steps. Moreover, every walk from $(0,0)$ to $(7,4)$ can be encoded as a series of right steps (R) and up steps (U). For instance the path $RRRUURRURRU$ corresponds to the following path.



So the number of walks is the same as the number of ways to arrange seven Rs and four Us which is $\binom{11}{4} = \binom{11}{7} = 330$ (i.e., choose when we take an up step or choose when we take a right step).

Example: On the square lattice shown below how many different walks are there from $(0,0)$ to $(7,4)$ which consists of steps to the right by one unit or up one unit?



Solution: This problem is nearly the exact same as the problem before except now we have to forbid some walks. In particular we have to throw out all walks that pass through the point $(4,2)$. So let us count how many walks there are that pass through $(4,2)$. Such a walk can be broken into two parts, namely a walk from $(0,0)$ to $(4,2)$ (of which there are $\binom{6}{2}$ such walks) and a walk from $(4,2)$ to $(7,4)$ (of which there are $\binom{5}{2}$ such walks). Since we can combine these two halves of the walk arbitrarily then by the Rule of Multiplication the number of walks that pass through $(4,2)$ is $\binom{6}{2}\binom{5}{2} = 150$. Therefore the number of walks not passing through $\binom{4}{2}$ is $330 - 150 = 180$.