

Student name: _____

Student ID: _____

MATH 61 (Butler)
Midterm II, 12 November 2008

This test is closed book and closed notes, with the exception that you are allowed one $8\frac{1}{2}'' \times 11''$ page of handwritten notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified. Each problem is worth 10 points.

1. There is a huge bin filled with many balls of many different colors. What is the smallest number n of balls you need to take from the bin so that among the n balls you *must* have either seven balls of one color or seven balls with no two the same color.

(Your answer needs two parts, first show that with n balls you must satisfy the condition, second show that with $n - 1$ balls you might not satisfy the condition.)

We can draw out 36 balls and not have what we need by drawing six balls in each of six colors. So 36 balls cannot guarantee a solution.

If we draw out 37 balls then we group them by colors. If there are seven or more colors then we must have seven balls no two of which are the same color. If there are six or fewer then by the pigeon hole principle some color grouping must have at least seven balls. Either way we have an acceptable situation and so 37 is enough to guarantee a solution.

1	
2	
3	
4	
5	
Σ	

2. Consider a variation of the tower of Hanoi problem where we have three poles A , B and C and start with a stack of n different sized discs on pole A arranged going from the bottom to the top by largest to smallest. We again consider the problem of moving the discs from pole A to pole C . There are three rules:

- (i) We can only move one disc at a time.
- (ii) We can never put a larger disc over a smaller disc.
- (iii) We can not move a disc directly from A to C or from C to A .

So for example, if $n = 1$ we now need two moves to transfer from pole A to pole C , first move from A to B and second move from B to C .

(a) Let R_n be the minimal number of moves needed to move all n discs from pole A to pole C . Set up a recurrence for R_n , explain your reasoning behind the recurrence.

Proceeding as in the tower of Hanoi problem we think about how to move a stack of n discs. We can break it into the following five steps:

1. Move the top $n - 1$ discs from A to C (using R_{n-1} moves).
2. Move the bottom disc from A to B (one move).
3. Move the top $n - 1$ discs from C to A (using R_{n-1} moves).
4. Move the bottom disc from B to C (one move).
5. Move the top $n - 1$ discs from A to C (using R_{n-1} moves).

Adding it up we have

$$R_n = R_{n-1} + 1 + R_{n-1} + 1 + R_{n-1} = 3R_{n-1} + 2.$$

(b) Solve the recurrence in part (a).

This is a constant coefficient linear *nonhomogeneous* recurrence relation. Substituting $R_n = a_n + B$ we get that

$$a_{n+1} + B = 3(a_n + B) + 2 = 3a_n + (3B + 2).$$

Choosing $B = -1$ this will reduce it to $a_{n+1} = 3a_n$ which has solutions of the form $a_n = C3^n$. So we have that $R_n = C3^n - 1$. Since $R_1 = 2 = 3C - 1$ we see that $C = 1$ and so

$$R_n = 3^n - 1.$$

3. Use the graph K_{m+n} to give a combinatorial proof that

$$\binom{m+n}{2} = \binom{m}{2} + mn + \binom{n}{2}.$$

Make sure to fully explain what you are counting on both sides.

We count the number of edges in K_{m+n} .

We know that the number of edges in the complete graph K_{m+n} is $\binom{m+n}{2}$ which gives us the left hand side.

To get the right hand side we break the vertices into two sets, one of size n and the other of size m . The edges now are in one of three places, in the complete graph corresponding to the set of size n (of which there are $\binom{n}{2}$ edges), the complete bipartite graph connecting these two sets (of which there are mn edges), or in the complete graph corresponding to the set of size m (of which there are $\binom{m}{2}$ edges). Adding these up give the right hand side.

4. Solve the recurrence relation

$$a_n = 2\sqrt{(a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2})} \quad \text{for } n \geq 2,$$

with initial conditions $a_0 = 1$, $a_1 = 2$.

We first square both sides to get

$$a_n^2 = 4(a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2}) = 4a_{n-1}^2 - 4a_{n-2}^2.$$

Substitute $b_n = a_n^2$ to turn this into the linear homogeneous recursion

$$b_n = 4b_{n-1} - 4b_{n-2}.$$

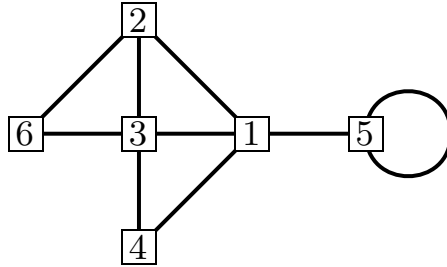
This becomes $r^2 - 4r + 4 = 0$ or $(r - 2)^2 = 0$ so that we have repeated roots $2, 2$. So we have that

$$a_n^2 = b_n = A2^n + Bn2^n \quad \text{or} \quad a_n = \sqrt{A2^n + Bn2^n}.$$

Initial conditions give us $1 = \sqrt{A}$ so that $A = 1$ and $2 = \sqrt{2 + 2B}$ so that $B = 2$. Thus our final solution is

$$a_n = \sqrt{(n + 1)2^n}.$$

5. A graph has an Eulerian cycle if there is a closed walk which uses each edge *exactly* once. A related problem is to find the shortest closed walk (i.e., using the fewest number of edges) which uses each edge *at least* once. (This is known as the “Chinese Postman” problem and comes up frequently in applications for optimal routing.) Consider the following graph on six vertices.



(a) Find a walk that starts and ends at vertex 1 of length 11 so that each edge in the graph is used at least once.

One such path is

$(1, 5, 5, 1, 2, 6, 3, 4, 1, 3, 2, 1).$

(b) Explain why 11 is the smallest number of edges needed to find a walk that starts and ends at vertex 1 and uses each edge at least once.

There are 9 edges. So we only need to show that two of the edges must be used twice.

Since we must go to 5 and return from 5 the edge connecting 1 and 5 must be used twice.

Consider the vertex 2, this has degree 3 and so we must visit it more than once (a first visit can only use two of the three edges). After a second visit we will have used four edges incident to 2 but there are only three edges so one edge must be used twice.