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## Midterm 2 Review Solutions

### Binomial coefficients

1. Using the binomial theorem with  $a = -1$  and  $b = 1$  we have

$$0 = 0^n = (-1 + 1)^n = \sum_{k=0}^n (-1)^k \binom{n}{k}.$$

This can be rewritten as

$$0 = \sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k},$$

or

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}.$$

The left hand side is the number of subsets with an even number of elements while the right hand side is the number of subsets with an odd number of elements establishing the result.

2. The binomial theorem gives us

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

We now integrate both sides to get

$$\int_{-1}^0 (1 + x)^n dx = \frac{(1 + x)^{n+1}}{n + 1} \Big|_{x=-1}^{x=0} = \frac{1}{n + 1}$$

$$\begin{aligned} \int_{-1}^0 \left( \sum_{k=0}^n \binom{n}{k} x^k \right) dx &= \sum_{k=0}^n \binom{n}{k} \left( \int_{-1}^0 x^k dx \right) \\ &= \sum_{k=0}^n \binom{n}{k} \frac{x^{k+1}}{k + 1} \Big|_{-1}^0 = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k + 1} \end{aligned}$$

Giving the result.

3. (a) By the binomial theorem we let  $a = 2$  and  $b = 1$  to get

$$3^n = (2 + 1)^n = \sum_{k=0}^n \binom{n}{k} 2^k 1^{n-k},$$

establishing the result.

(b) We give two ways to count the number of 2-colored subsets. First every subset corresponds to a ternary word of length  $n$ , i.e., a word composed of  $\{0, 1, 2\}$  where in the  $i$ th slot 0 indicates  $i$  is not included, 1 indicates  $i$  is included with color 1 and 2, indicates  $i$  is included with color 2. There are  $3^n$  such words and each word corresponds to exactly one set and each set corresponds to one word so there are  $3^n$  2-colored subsets.

On the other hand we can count them by first taking a subset with  $k$  elements and assigning each element in that subset one of 2 colors, which can be done in  $2^k$  ways. So the number of  $k$  element 2-colored subsets is  $2^k \binom{n}{k}$  and finally by the addition rule we add up the possibilities for  $k = 0, 1, \dots, n$  to get the number of 2-colored subsets is  $\sum_{k=0}^n 2^k \binom{n}{k}$ .

4. We have that

$$T_n = n + (n - 1) + \dots + 1 = \frac{n(n + 1)}{2} = \binom{n + 1}{2}.$$

5. We have that

$$\begin{aligned} \Delta_n &= T_n + T_{n-1} + \dots + T_1 \\ &= \binom{n + 1}{2} + \binom{n}{2} + \dots + \binom{2}{2} \\ &= \binom{n + 1}{2} + \binom{n}{2} + \dots + \binom{2}{2} + \binom{1}{2} + \binom{0}{2} \\ &= \binom{n + 2}{3}. \end{aligned}$$

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### Pigeon hole principle

1. Let  $\{a_1, \dots, a_{13}\}$  be the chosen numbers. Now consider the numbers

$$a_1, \dots, a_{13}, a_1 + 5, \dots, a_{13} + 5.$$

These 26 numbers lie in the range between 1 and 25 and so by the pigeon hole principle there are two numbers which match. The only way this can happen is if for some  $i$  and  $j$  we have  $a_i = a_j + 5$ , i.e., we have two terms which differ by five.

2. We similarly consider the numbers

$$a_1, \dots, a_{13}, a_1 + 6, \dots, a_{13} + 6.$$

These 26 numbers lie in the range between 1 and 26, so we cannot use the pigeon hole principle. But now suppose that there was no two which differed by six, then all of the numbers in our list would have to be distinct, so our set *must* have the numbers 1, 2, 3, 4, 5, 6 and the numbers 15, 16, 17, 18, 19, 20. But we still need one more number, and there is no remaining number that we can choose that does not differ by six from one of these. So this is impossible. Therefore there must be two which differ by six.

3. Such a set is  $\{1, 2, 3, 4, 5, 6, 7, 15, 16, 17, 18, 19, 20\}$ .
4. There are four people so suppose that we have 23 gumballs of the first color, and 5 for each of the remaining colors. This gives us 43 gumballs and there is no way that everyone can have 6 of

a single color. So 43 is not enough to guarantee that we can do it.

Now suppose that there are 44 gumballs. By the pigeon hole principle some color has at least 9 gumballs of that color so we take out six and give it to the first friend. There are now 38 gumballs and by the pigeon hole principle some color has at least 8 gumballs of that color so we take out six and give it to the second friend. There are now 32 gumballs and by the pigeon hole principle some color has at least 7 gumballs of that color and so we take out six and give it to the third friend. There are now 26 gumballs and by a last application of the pigeon hole principle some color must have at least 6 gumballs of that color and you take those six. Now everyone has six gumballs that are all the same color.

So  $n = 44$ .

### Recursion

1. First note that  $q_2 = q_3 = 1$ ,  $q_4 = 2$  (since  $4 = 2 + 2$ ), and  $q_5 = 3$  (since  $5 = 2 + 3 = 3 + 2$ ). In general, for  $n \geq 4$  we can look at the last term in the composition, it can be either  $2, 3, \dots, n - 2, n$  (1 and  $n - 1$  are not allowed because each part must be  $\geq 2$ ). The last term being  $n$  contributes 1 to the count, while the remaining  $i$  will contribute  $q_i$  to the count giving

$$q_n = 1 + q_2 + q_3 + \dots + q_{n-2}.$$

Now taking the difference between  $q_n$  and  $q_{n-1}$  we have for  $n \geq 5$  that

$$\begin{aligned} q_n - q_{n-1} &= (1 + q_2 + \dots + q_{n-2}) - (1 + q_2 + \dots + q_{n-3}) \\ &= q_{n-2} \end{aligned}$$

or  $q_n = q_{n-1} + q_{n-2}$ . We have that  $q_2 = F_1$ ,  $q_3 = F_2$ ,  $q_4 = F_3$  and for  $n \geq 5$  that  $q_n$  satisfies the same recursion as  $F_n$  and so  $q_n = F_{n-1}$ .

2. For  $n \geq 4$  we consider how an  $n$ -bit string can end. We have one of three possibilities, either 1, 10 or 100, i.e.,

$$\underbrace{* \dots *}_{n-1} 1, \quad \underbrace{* \dots *}_{n-2} 10, \quad \underbrace{* \dots *}_{n-3} 100.$$

The remaining “ $* \dots *$ ” can be filled in with any string of the appropriate length which does not have a 000 and so we get the recursion

$$w_n = w_{n-1} + w_{n-2} + w_{n-3}.$$

We now need to compute the first few terms to get the recursion started. We have  $w_1=2$  (1

or 0),  $w_2=4$  (11, 10, 01 or 00) and  $w_3=7$  (all but 000). We can now compute the next few terms to get  $w_4=2+4+7=13$ ,  $w_5=4+7+13=24$ ,  $w_6=7+13+24=44$  and  $w_7=13+24+44=81$ .

3. Our only hope is to find a pattern. So we begin

$$\begin{aligned} d_3 &= \frac{1 + d_2}{d_1} = \frac{1 + y}{x} \\ d_4 &= \frac{1 + d_3}{d_2} = \frac{1 + (1 + y)/x}{y} = \frac{1 + x + y}{xy} \\ d_5 &= \frac{1 + d_4}{d_3} = \frac{1 + (1 + x + y)/xy}{(1 + y)/x} \\ &= \frac{1 + x + y + xy}{y(1 + y)} = \frac{(1 + x)(1 + y)}{(1 + y)y} = \frac{1 + x}{y} \\ d_6 &= \frac{1 + d_5}{d_4} = \frac{1 + (1 + x)/y}{(1 + x + y)/xy} \\ &= \frac{x(1 + x + y)}{1 + x + y} = x \\ d_7 &= \frac{1 + d_6}{d_5} = \frac{1 + x}{(1 + x)/y} = y \end{aligned}$$

Note that  $d_1 = d_6$  and  $d_2 = d_7$ , in particular this repeats every five times, i.e.,  $d_n = d_{n+5}$ . Now to compute  $d_{5583975}$  we note that this is a multiple of 5 and so we have

$$d_{5583975} = d_5 = \frac{1 + x}{y}.$$

### Solving recursions

1. This is a linear homogeneous recurrence with constant coefficients of order 3, so we first turn it into a polynomial equation.

$$r^3 = 6r^2 - 11r + 6 \quad \text{or} \quad r^3 - 6r^2 + 11r - 6 = 0.$$

This is a cubic, but in this case it is not too hard to see that 1 is a root so we can factor it out to get

$$\begin{aligned} 0 &= r^3 - 6r^2 + 11r - 6 \\ &= (r - 1)(r^2 - 5r + 6) \\ &= (r - 1)(r - 2)(r - 3). \end{aligned}$$

So the roots are 1, 2, 3. We now have that the general solution for the recursion is

$$r_n = A + B2^n + C3^n$$

for appropriate choices of  $A$ ,  $B$  and  $C$ . Plugging the initial conditions in we have

$$\begin{aligned} 5 &= A + B + C \\ 6 &= A + 2B + 3C \\ 10 &= A + 4B + 9C \end{aligned}$$

Subtracting the first from the second and third we can reduce this to

$$\begin{aligned} 5 &= A + B + C \\ 1 &= B + 2C \\ 5 &= 3B + 8C \end{aligned}$$

If we now multiply the second by three and subtract that from the third this further reduces to

$$\begin{aligned} 5 &= A + B + C \\ 1 &= B + 2C \\ 2 &= 2C \end{aligned}$$

We can now work our way back up. we clearly have  $C = 1$ , substituting this into the second equation we have  $B = -1$  and substituting both of these into the first equation we have  $A = 5$ . So our solution is

$$r_n = 5 - 2^n + 3^n.$$

To check this we note that

$$\begin{aligned} 5 - 2^3 + 3^3 &= 5 - 8 + 27 = 24 \\ &= 60 - 66 + 30 = 6r_2 - 11r_1 + 6r_0 \end{aligned}$$

so that it does give us the next correct term. (This does not guarantee that we are correct but does give us an indication that we are probably correct.)

2. This is a linear homogeneous recurrence with constant coefficients of order 2, so we first turn it into a polynomial equation.

$$r^2 = 2r + 1 \quad \text{or} \quad r^2 - 2r - 1 = 0.$$

This is a quadratic and we can use the quadratic equation to solve for the roots. Doing so we get

$$r = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2}.$$

We now have that the general solution for the recursion is

$$P_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n$$

for appropriate choices of  $A$  and  $B$ . Plugging the initial conditions in we have

$$\begin{aligned} 0 &= A + B, \quad \text{and} \\ 1 &= A(1 + \sqrt{2}) + B(1 - \sqrt{2}) \\ &= (A + B) + (A - B)\sqrt{2}. \end{aligned}$$

The first equation gives us  $B = -A$  substituting this into the second equation we have that

$2A\sqrt{2} = 1$  or  $A = 1/(2\sqrt{2})$  and so  $B = -1/(2\sqrt{2})$ . So our solution is

$$P_n = \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^n - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^n.$$

As before we can check this. First we have that  $P_2 = 2P_1 + P_0 = 2$  while we also have

$$\begin{aligned} P_2 &= \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^2 - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^2 \\ &= \frac{1 + 2\sqrt{2} + 2}{2\sqrt{2}} - \frac{1 - 2\sqrt{2} + 2}{2\sqrt{2}} = \frac{4\sqrt{2}}{2\sqrt{2}} = 2 \end{aligned}$$

which agrees with what it should be.

3. This is not a linear homogeneous recurrence with constant coefficients. But perhaps we can turn it into one. First note that we can multiply through by  $n$  and rewrite this as

$$nS_n = (n - 1)S_{n-1} + 2.$$

So here there is an obvious substitution,  $b_n = nS_n$ , our initial condition then becomes  $b_2 = 2$ . Then this recurrence becomes

$$b_n = b_{n-1} + 2.$$

Along with  $b_2 = 2$  it is easy to see that this recurrence gives  $b_n = 2n - 2$ . So we have  $nS_n = 2n - 2$  or

$$S_n = 2 - \frac{2}{n}.$$

Again we can check that  $S_3 = 2 - (2/3) = 4/3$  but also

$$S_3 = S_2 + \frac{2 - S_2}{3} = 1 + \frac{1}{3} = \frac{4}{3},$$

so both agree.

4. Following the hint we notice that we alternately add  $2, 0, 2, 0, 2, 0, \dots$ . This can be rewritten as adding  $1 + (-1)^n$  so that our recurrence is

$$t_n = t_{n-1} + 2t_{n-2} + 1 + (-1)^n.$$

So we now have a non-homogeneous linear recurrence with constant coefficients. So we will solve this by letting

$$t_n = s_n + g(n),$$

where  $g(n)$  is some appropriate function of  $n$  (which we will soon determine) and  $s_n$  satisfies the homogeneous linear recurrence

$$s_n = s_{n-1} + 2s_{n-2}.$$

This part can easily be solved. We have

$$r^2 = r + 2 \quad \text{or} \quad r^2 - r - 2 = 0.$$

This quadratic factors as  $(r-2)(r+1) = 0$  so that the roots are 2 and  $-1$ . So we now have

$$s_n = A2^n + B(-1)^n,$$

where  $A$  and  $B$  are constants that we will determine later. We are now ready to guess our form for  $g(n)$ . We would normally guess  $g(n) = C + D(-1)^n$ , but since  $-1$  is also a solution to the homogeneous equation we need to modify our guess to  $g(n) = C + Dn(-1)^n$ . So we have

$$t_n = A2^n + B(-1)^n + C + Dn(-1)^n$$

First we will determine  $C$  and  $D$  by putting  $t_n$  into our recurrence

$$\begin{aligned} & A2^n + B(-1)^n + C + Dn(-1)^n \\ &= (A2^{n-1} + B(-1)^{n-1} + C + D(n-1)(-1)^{n-1}) \\ &+ 2(A2^{n-2} + B(-1)^{n-2} + C + D(n-2)(-1)^{n-2}) \\ &\quad + 1 + (-1)^n \end{aligned}$$

Collecting and removing any terms which cancel we have

$$0 = (2C + 1) + (-1)^n(-3D + 1)$$

So that we need to choose  $C = -1/2$  and  $D = 1/3$ . We now have

$$t_n = A2^n + B(-1)^n - \frac{1}{2} + \frac{1}{3}n(-1)^n$$

we now need to determine  $A$  and  $B$  which can be done by using the initial conditions.

$$\begin{aligned} 0 &= A + B - \frac{1}{2} \\ 0 &= 2A - B - \frac{5}{6} \end{aligned}$$

Adding the two equations we see that  $3A - 4/3 = 0$  or  $A = 4/9$  and

$$B = \frac{1}{2} - A = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

So our final answer is

$$t_n = \frac{4}{9}2^n + \left(\frac{6n+1}{18}\right)(-1)^n - \frac{1}{2}.$$

To check our answer we note that  $t_2 = t_0 + 2t_1 + 2 = 2$  while

$$\frac{4}{9}2^2 + \left(\frac{13}{18}\right)(-1)^2 - \frac{1}{2} = \frac{32}{18} + \frac{13}{18} - \frac{9}{18} = \frac{36}{18} = 2.$$

5. This is a linear homogeneous recurrence with constant coefficients of order 2, so we first turn it into a polynomial equation

$$r^2 = 4r - 4 \quad \text{or} \quad r^2 - 4r + 4 = 0.$$

This factors as  $(r-2)^2 = 0$  and so we have a repeated root of 2. So we have that the general solution is

$$c_n = A2^n + Bn2^n$$

for appropriate choice of  $A$  and  $B$  (note that in the second term we added a factor of “ $n$ ” to account for the fact that this is a repeated root). To find  $A$  and  $B$  we use the boundary conditions.

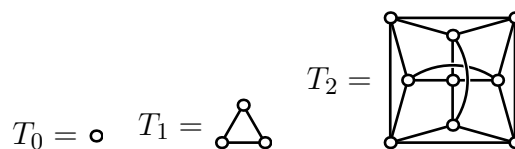
$$\begin{aligned} 5 &= A \\ 17 &= 32A + 160B \end{aligned}$$

From this we see that  $A = 5$  and putting this into the second equation that  $17 = 160 + 160B$ , or  $160B = -143$  or  $B = -143/160$ . So the general solution is

$$c_n = 5 \cdot 2^n - \frac{143}{160}n2^n.$$

## Graphs

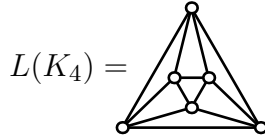
- To show that a graph is bipartite we have to show how to divide the vertices into two sets so that all edges connect vertices between these two sets. Recall that in a hypercube that the edges connect two binary strings which differ in one entry. So to go from one vertex to another vertex we either change a single 1 to a 0 or a single 0 to a 1. In either case we are changing the number of 1s by 1, and in particular a vertex with an odd number of 1s connects to vertices with an even number of 1s and vice versa. So the two groups of vertices are those which have an even number of 1s and those which have an odd number of 1s.
- To make a trinary word we choose one of three letters for the first slot, one of three letters for the second slot, ..., and one of three letters for the  $n$ th slot. So this gives us  $3^n$  trinary words and so  $T_n$  has  $3^n$  vertices. To count the number of vertices we note that at any vertex that to find a vertex that it is adjacent to, we pick one slot (out of  $n$ ) and then change that to a different number (which there are 2 possible ways to do), so each vertex is incident to  $2n$  edges. Adding up all the degrees and dividing by 2 we can conclude that the graph has  $n3^n$  edges.
- The graphs are shown below



4. The number of vertices in  $L(K_n)$  is the same as the number of edges in  $K_n$  which is  $\binom{n}{2}$ . For a given edge  $e = \{u, v\}$  in  $K_n$  there are an additional  $n - 2$  edges incident at  $u$  and  $n - 2$  edges incident at  $v$  and so in  $L(K_n)$  the corresponding vertex will be adjacent to  $(n-2) + (n-2) = 2n-4$  edges. Adding up over all the degrees and dividing by 2 we have that the total number of edges is

$$(n-2) \binom{n}{2}.$$

A graph of  $L(K_4)$  is shown below.

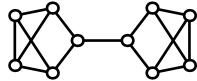



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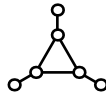
### Eulerian graphs

1. Suppose that the graph does have a bridge, let us remove that edge and we now split it into two components. Each component now has one vertex of degree  $r - 1$  (which is odd, this corresponds to the vertex with the bridge) and the remaining vertices have degree  $r$  (which is even). But this is impossible because in any component there must be an even number of vertices of odd degree. So it must be the case that there is no bridge.

2. One example is shown below.



3. (a) This is impossible because the graph would have an *odd* number of odd vertices.  
 (b) This is possible, in fact there is only one such graph and it is shown below.



- (c) This is impossible because the two vertices of degree 5 would have to connect to every other vertex and so the minimum degree of the remaining vertices is at least 2, in particular there is no way we can get the vertex of degree 1.  
 4. In a bipartite graph we can split the vertices into two disjoint sets  $U$  and  $W$  so that edges only go between  $U$  and  $W$ . If we are in a walk then the vertices must alternate between  $U$  and  $W$ . In particular, if we start at a vertex in  $U$  then after an odd number of steps we are in  $W$  and after an even number of steps we are in  $U$ ; in order to be

closed we must end back in  $U$  so must take an even number of steps. Similarly if we had started at a vertex in  $W$ .

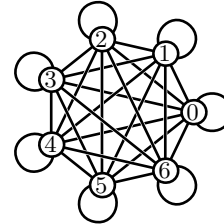
5. Fix a vertex  $v$  and for every other vertex  $w$  fix a walk connecting  $v$  to  $w$ . We now split the vertices of the graph into two sets  $V_e$  and  $V_o$  by putting a vertex  $w$  in  $V_e$  if the walk connecting  $v$  to  $w$  has even length and putting a vertex  $w$  in  $V_o$  if the walk connecting  $v$  to  $w$  has odd length.

Suppose that there is an edge connecting two vertices  $p, q$  of  $V_e$  then we can take the walk starting at  $v$  going to  $p$  then add the edge to  $q$  and then take the walk backwards between  $q$  and  $v$ . This is a closed walk from  $v$  to  $v$  which has *odd* length. But that is impossible since the graph has no odd closed walks. Similarly there are no edges connecting vertices  $p, q$  of  $V_o$ . So all edges of the graph go between  $V_e$  and  $V_o$  and so the graph is bipartite.

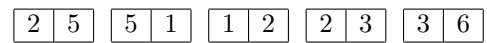
6. We first note that we can relate dominoes in a complete set to the edges of the complete graph with loops at each vertex (see the picture below), i.e., the edge  $\{2, 5\}$  corresponds to the domino 

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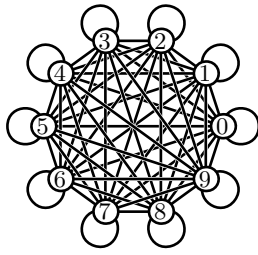


The next thing to note is that a walk in the closed graph can be turned into a placement of dominoes with adjacent dominoes oriented so that the touching squares have the same number of pips. For example, the walk  $(2, 5, 1, 2, 3, 6)$  translates into the following sequence of dominoes:



In particular we need to find a walk in the graph that uses each edge (i.e., domino) exactly once that starts and ends at the same vertex. This is precisely an Eulerian cycle in the graph! Since all the vertices of this graph have degree six, which is even, such a cycle and so such a configuration exists.

7. Continuing from the previous problem we have the same setup as before but now our graph is the following.



In this graph all the vertices have degree nine, which is odd, and so this graph has no Eulerian cycle and so it is impossible to arrange the dominoes in the required circular manner.

8. We need to remove dominoes so that the graph in the previous problem has vertices of all even degree. There are ten vertices and each edge can effect the degree of two vertices so we need to remove at least five edges. On the other hand five is certainly sufficient since we can remove the dominoes  $\boxed{0 \mid 1}$ ,  $\boxed{2 \mid 3}$ ,  $\boxed{4 \mid 5}$ ,  $\boxed{6 \mid 7}$ , and  $\boxed{8 \mid 9}$ . Which will leave a graph with all vertices of degree eight so is now Eulerian and using the same techniques as in the previous problem we can arrange the dominoes.
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