Midterm 2 Review

(Test on 12 November 2008)

Binomial coefficiencts

The binomial coefficient $\binom{n}{k}$ is used to count the number of k-element subsets of an n-element set. The term binomial coefficient comes from the binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Various properties involving binomial coefficients can be proved by using the above relationship for appropriate choices of a and b. Binomial coefficients satisfy some important properties:

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

The binomial coefficients can be arranged in a triangular pattern known as Pascal's triangle. The first few rows of Pascal's triangle are given below.

$\binom{n}{k}$	k=0	k=1	k=2	k=3	k=4	k=5	k=6
n=0	1						
n=1	1	1					
n=2	1	2	1				
n=3	1	3	3	1			
n=4	1	4	6	4	1		
n=5	1	5	10	10	5	1	
n=6	1	6	15	20	15	6	1

A *combinatorial proof* is used to show that "$A = B$" by counting some collection of objects in two different ways. For example, there are 2^n subsets of an n-element set but there are also $\binom{n}{k}$ subsets with exactly k elements since the possible size of a subset is $0, 1, \ldots, n$ we have by the addition rule that there are $\sum_{k=0}^{n} \binom{n}{k}$ subsets; combining we have $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

Examples:

1. Show for $n \geq 1$ that

   $$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

   Use this to show that the number of subsets with an *even* number of elements is equal to the number of subsets with an *odd* number of elements for $n \geq 1$.

2. Show for $n \geq 0$ that

   $$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} = \frac{1}{n+1}.$$

   (Hint: use integration between -1 and 0 on some form of the binomial equation.)

3. Give a proof for the following identity for $n \geq 0$

   $$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

   (a) by using the binomial theorem.

   (b) by counting the number of 2-colored subsets of an n-element set. (A 2-colored subset is a subset where each element has been assigned one of two colors, so for instance there are 9 2-colored subsets of $\{1, 2\}$, they are $\emptyset$, $\{\overline{1}\}$, $\{\underline{1}\}$, $\{\overline{2}\}$, $\{\underline{2}\}$, $\{\overline{1}, \overline{2}\}$, $\{\overline{1}, \underline{2}\}$, $\{\underline{1}, \overline{2}\}$ and $\{\underline{1}, \underline{2}\}$.

4. The triangular number T_n is the number of tennis balls needed to form a triangle with n balls in the first row, $n-1$ balls in the second row, …, and 1 ball in the top row. Find an expression for T_n in terms of binomial coefficients.

5. The tetrahedron number Δ_n is the number of tennis balls needed to form a tetrahedron with T_n balls in the first later, T_{n-1} balls in the second layer, …, and T_1 balls in the top layer. Find an expression for Δ_n in terms of binomial coefficients.

Pigeon hole principle

The *pigeon hole principle* says that if you are putting r objects into n different groups then one of the groups has at least $\lceil r/n \rceil$ objects. This is used to show that two objects must be similar. Note that while this can be used to show that there are two or more objects which are in the same group we cannot say which group has multiple objects.

Proof by *contradiction* is a powerful technique where we show that something is true by first assuming the opposite holds and then end up with a contradiction (i.e., showing that the opposite cannot hold).

Examples:

1. If we pick 13 numbers between 1 and 20 show that there are two which differ by exactly 5.

2. If we pick 13 numbers between 1 and 20 show that there are two which differ by exactly 6.

3. Show it is possible to pick 13 numbers between 1 and 20 so that no two differ by exactly 7.

4. You and your three friends see a gumball machine which has five different colors of gumballs. You decide to keep buying gumballs until everyone has six gumballs of a single color (it is allowed that two different people have the same color of gumballs). What is the smallest number n of gumballs you will need to buy in order to guarantee that everyone gets their gumballs?

   (Your answer will consist of two parts, first show that you can always satisfy the condition with n

gumballs, while with $n-1$ it is possible that you might not satisfy the condition.)

Recursion

A recurrence relationship on a sequence a_n is a relationship between a_n and the previous terms, i.e., a_i for $i < n$. An example is the minimal number of moves needed to move n discs in the Tower of Hanoi problem t_n, by moving the top $n-1$ discs, then the bottom disc and the top $n-1$ discs again, so$t_n = 2t_{n-1} + 1$.

An important sequence is the *Fibonacci numbers* which are defined recursively by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ (i.e., to get the next Fibonacci number add the two most recent ones). This gives the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

Examples:

1. Let q_n be the number of compositions (i.e., write n as a sum of ordered nonnegative integers, or "part") of n with each part ≥ 2. For example $q_6 = 5$ since $6 = 4+2 = 2+4 = 3+3 = 2+2+2$ Give a recurrence for q_n. Show $q_n = F_{n-1}$ for $n \geq 2$.

2. Let w_n be the number of binary sequences with no 000. Find a recurrence for w_n. Give the numbers $w_1, w_2, \ldots, w_7$. These numbers are known as the *tribonacci* numbers.

3. Consider the recurrence relationship

$$d_{n+2} = \frac{1 + d_{n+1}}{d_n}, \quad \text{for } n \geq 1.$$

   Given $d_1{=}x$ and $d_2{=}y$ with $x, y > 0$ find $d_{5583975}$ in terms of x and y.

Solving recursions

To solve a recurrence is to find an explicit expression for a_n which depends only on n (i.e., you do not need to know any of the previous terms). For example in the Tower of Hanoi problem we have the recurrence $t_n = 2t_{n-1} + 1$ and initial condition $t_1 = 1$, this has solution $t_n = 2^n - 1$.

We can solve recurrences by either looking for a pattern and then *verifying* our guess using induction, i.e., showing that our guess satisfies the initial conditions and the recursion relation; or by a systematic method. The latter case applies for *linear homogeneous recurrence relations with constant coefficients of order* k, i.e., a recursion which can be written as

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, \ldots, c_k$ are constants (i.e., independent of n). The method to solve these is to first translate this into a polynomial

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k.$$

Then solve for the roots $r_1, r_2, \ldots, r_k$ of this polynomial, either by factoring or the quadratic equation. If these roots are distinct then

$$a_n = D_1 r_1^n + D_2 r_2^n + \cdots + D_k r_k^n,$$

where $D_1, D_2, \ldots, D_k$ are constants which are determined by the initial conditions. In the case of repeated roots the process is similar except now instead of using $D_1 r_1^n + D_2 r_1^n + \cdots + D_\ell r_1^\ell$ we introduce powers of n to distinguish solutions so we have $D_1 r_1^n + D_2 n r_1^n + \cdots + D_\ell n^{\ell-1} r_1^\ell$

In the case that we have a non-homogeneous term such as

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + d n^\ell r^n$$

then we can substitute

$$a_n = b_n + E_0 r^n + E_1 n r^n + \cdots + E_\ell n^\ell r^n$$

where the b_n will handle the homogeneous part and the constants $E_0, E_1, \ldots, E_\ell$ are chosen to handle the non-homogeneous part. Note in the case that r is also part of the homogeneous solution we need to modify our guess appropriately by multiplying the non-homogeneous terms of our substitution by n.

Occasionally we can use substitution to transform a recursion into something which we can use the above techniques on. Finally, if nothing else works we can always look at the first few terms and try to find a pattern.

Examples:

1. Solve the recurrence relationship

$$r_{n+3} = 6r_{n+2} - 11r_{n+1} + 6r_n, \quad \text{for } n \geq 0$$

   with initial conditions $r_0 = 5$, $r_1 = 6$ and $r_2 = 10$.

2. Solve the recurrence relationship

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{for } n \geq 2$$

   with initial conditions $a_0 = 0$ and $a_1 = 1$. These numbers are known as the *Pell numbers*.

3. Solve the recurrence relationship

$$S_n = S_{n-1} + \frac{2 - S_{n-1}}{n}, \quad \text{for } n \geq 3$$

   with initial condition $S_2 = 1$.

4. Solve the recurrence relationship

$$t_n = \begin{cases} t_{n-1} + 2t_{n-2} + 2 & \text{if } n \text{ is even} \\ t_{n-1} + 2t_{n-2} & \text{if } n \text{ is odd} \end{cases}$$

   with initial conditions $t_0 = 0$ and $t_1 = 0$. (Hint: try to rewrite the recurrence into a single expression.)

5. The initial conditions are not always consecutive. Sometimes we might now information about the behavior at the ends (or boundary) and we are trying to extrapolate information about what is happening between. Solve the following recursion

$$c_n = 4c_{n-1} - 4c_{n-2}, \quad \text{for } n \geq 1$$

with boundary conditions $c_0 = 5$, and $c_5 = 17$.

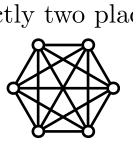
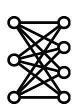
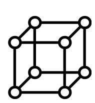
Graphs

A graph $G = (V, E)$ consists of two sets, a vertex set V and an edge set E. The edge set either satisfies $E \subseteq V^{(2)}$ (i.e., two-element subsets where order does not matter so edges are not directed) or $E \subseteq V \times V$ (i.e., two-element lists where order does matter so edges are directed). Pictorially vertices are represented by points "o" and edges by lines "o—o"(when it is undirected) or arrows "o→o" (when it is directed).

Graphs can be used to model many interactions (we already used directed graphs to model relations). A simple graph is an undirected graph without loops (an edge that goes from a vertex and returns to itself) or parallel edges (two or more edges going between the same vertices).

A graph $G = (V, E)$ is bipartite if we can split V into two disjoint sets $V = V_1 \cup V_2$ so that all edges connect a vertex in V_1 to a vertex in V_2.

Some important simple graphs are the complete graph on n vertices, K_n, which has n vertices and all $\binom{n}{2}$ possible edges; the complete bipartite graph $K_{m,n}$ which has $m + n$ vertices in two parts, one part with m vertices and the other with n vertices and all mn possible edges between these two parts; the hypercube H_n or Q_n with vertices all possible 2^n binary strings and the $n2^{n-1}$ edges connect two strings which differ in exactly two places.

$K_6 =$  $K_{3,4} =$  $Q_3 =$ 

Examples:

1. Show that the hypercube Q_n is bipartite.

2. Consider a graph T_n where vertices are "trinary" words of length n (i.e., a string of length n using the letters $\{0, 1, 2\}$) and edges connect two words which differ in one entry. How many vertices does T_n have? How many edges does T_n have?

3. Continuing from the previous exercise, draw T_0, T_1 and T_2.

4. Given a graph $G = (V, E)$, the line graph $L(G) = (E, F)$ is the graph where each *edge* of G is a *vertex* of $L(G)$ and two vertices in $L(G)$ are adjacent if the corresponding edges are incident to the same vertex in G. How many vertices are in $L(K_n)$? How many edges are in $L(K_n)$? Draw $L(K_4)$.

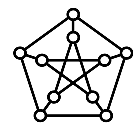
Eulerian graphs

Two vertices are *adjacent* if there is an edge connecting them, an edge and a vertex are *incident* if the edge connects to the vertex. In an undirected graph the degree of a vertex v, denoted $\delta(v)$ or $d(v)$, is the number of edges incident to v (loops count twice). An important fact about the degrees is

$$\sum_{v \in V} d(v) = 2(\# \text{ of edges}).$$

One consequence of this is that there must be an even number of vertices with odd degree.

A graph is *regular* if all vertices have the same degree. One famous example of a regular graph is the Petersen graph on ten vertices which is regular of degree three (shown below).



A *walk* (or in some books a path) of length n is a sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that v_{i-1} is adjacent to v_i for $i = 1, \ldots, n$. We say the walk is closed if it also satisfies $v_0 = v_n$. A graph is *connected* if between any two vertices there is a path connecting them.

A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. An *induced subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ where all possible edges are taken. The *components* of G are the maximal connected induced subgraphs (i.e., the connected pieces of G).

An Eulerian cycle on a graph is a closed walk that uses each edge of the graph *exactly* once. A graph has an Eulerian cycle if and only if it is connected and the degree of each vertex is *even*.

Examples:

1. A *bridge* in a connected graph is an edge whose removal makes the graph not connected. Show that if a graph is connected and regular of degree r, with r even, then the graph has no bridge.

2. Give an example of a connected graph which is regular of degree three which has a bridge.

3. Which of the following list of degrees is possible for a simple graph on six vertices. (If it is possible draw a graph that corresponds to it, while if it is not possible explain why it is not possible.)

   (a) $5, 4, 3, 2, 2, 1$

   (b) $3, 3, 3, 1, 1, 1$

(c) $5, 5, 4, 3, 2, 1$

4. Show that if a graph is bipartite all closed walks have *even* length.

5. Show that if all the closed walks in a graph have even length then the graph is bipartite. (You may assume that the graph is connected.)

6. A domino is a rectangle divided into two squares, each square having between 0 and 6 pips (it is possible for both squares to have the same number of pips). A full set of dominoes is one where each possible domino occurs once. Show that the dominoes can be arranged in a circular pattern so that for two adjacent dominoes the touching squares have the same number of pips.

7. Show that if the dominoes instead have between 0 and 9 pips on each square that it is impossible to arrange a full set of dominoes in a circular pattern so that for two adjacent dominoes the touching squares have the same number of pips.

8. What is the smallest number of dominoes that have to be removed from the complete set in the previous problem so that the remaining dominoes can now be arranged in a circular fashion?