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MATH 61 (Butler)

Midterm I, 20 October 2008

This test is closed book and closed notes, with the exception that you are allowed one $8\frac{1}{2}'' \times 11''$ page of handwritten notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified. You do not need to simplify terms such as "5!" or "$\binom{13}{5}$" (we would prefer if you didn't even try!). Each problem is worth 10 points.

1. Stan's Donuts is considering selling donuts in packs of three or seven (the "Baker's half-dozen"). Show, using induction, that if Stan's makes this change that for any number of donuts $k \geq 12$ it will always be possible to purchase k donuts.

For $k = 12$ we can buy four packs of three. For $k = 13$ we can buy two packs of three and a pack of seven. For $k = 14$ we can buy two packs of seven.

Now suppose that we know the statement is true up through k (and we can assume that $k \geq 14$). Now consider the case $k + 1$. By the induction hypothesis we know that we can purchase $(k + 1) - 3 = k - 2$ donuts (note that $k - 2 \geq 12$) so now add one more pack of three to form the purchase for $k + 1$.

1	
2	
3	
4	
5	
Σ	

2. Let $X = \{1, 2, 3, 4, 5\}$ and form the relationship R by aRb if $a + 2b$ is a multiple of 4.

(a) We have looked at three representations of relations; as a *set*, as a *graph*, and as a *matrix*. Produce two of the three representations for the relation R.

As a set:

$$R = \{(2, 1), (2, 3), (2, 5), (4, 2), (4, 4)\}.$$

As a matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Which of the following properties does this relation have: *reflexive*, *symmetric*, *anti-symmetric*, *transitive*, *partial order* and *equivalence*. (You do not need to justify your answer.)

The only property that the relationship satisfies is anti-symmetric.

3. For this problem we will consider the word "BOOKKEEPER".

(a) How many ways are there to rearrange the letters of the word BOOKKEEPER?

Grouping by letters we have 1-B, 2-O, 2-K, 3-E, 1-P, 1-R (a total of 10 letters). So the number of ways to rearrange is

$$\frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}.$$

(b) How many ways are there to rearrange the letters of the word BOOKKEEPER if no two E's can be adjacent. (For example EROOKKEBEP has no two adjacent E's so hould be counted but BEEPROKOKE has two adjacent E's so should not be counted.)

First we rearrange the remaining seven letters. There are then eight slots that the E's can go into and we choose three of them. This gives us

$$\frac{7!}{1!\,2!\,2!\,1!\,1!}\binom{8}{3}.$$

4. Let $X = \{1, 2, 3, 4, 5\}$, let $f : X \to X$ by $f(1) = 3, f(2) = 2, f(3) = 5, f(4) = 1, f(5) = 4$ and let $g : X \to X$ by $g(1) = 4, g(2) = 1, g(3) = 2, g(4) = 5, g(5) = 3$.

Is there a function $h : X \to X$ so that $h \circ f = g$ (i.e., so that for all x that $(h \circ f)(x) = g(x)$)? If there is a function, write one down. If not, prove there cannot be one.

If there were a function then we would need

$$\begin{aligned} h(3) &= (h \circ f)(1) = g(1) = 4 \\ h(2) &= (h \circ f)(2) = g(2) = 1 \\ h(5) &= (h \circ f)(3) = g(3) = 2 \\ h(1) &= (h \circ f)(4) = g(4) = 5 \\ h(4) &= (h \circ f)(5) = g(5) = 3 \end{aligned}$$

We now see that there is a function and we can define it by the following:

$$\begin{aligned} h(1) &= 5 \\ h(2) &= 1 \\ h(3) &= 4 \\ h(4) &= 3 \\ h(5) &= 2 \end{aligned}$$

5. The local Pizza Shack currently has a coupon where you can get either a two topping pizza plus a soda OR a three topping pizza without a soda for fifteen bucks. If there are 10 different types of toppings and 6 different types of soda how many different meal possibilities can you choose from if you use the coupon?
(Note: all the toppings on the pizza must be different.)

There are $\binom{10}{2}$ different two topping pizzas and 6 sodas so by the multiplication rule there are $\binom{10}{2}\cdot 6$ possibilities. For the second option there are $\binom{10}{3}$ different three topping pizzas. So combining using the addition rule we have

$$\binom{10}{2}\cdot 6 + \binom{10}{3}.$$