Midterm 1 Review Solutions

Induction

1. Done in class.
2. Base case: $1^2 = 1 = 1 \cdot 2 \cdot 3/6$.

   Induction step: Assume it is true for $k \leq n$ and consider the case $n+1$, we have

   $$\begin{aligned} &1^2 + 2^2 + \cdots + n^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left(\frac{2n^2+n}{6} + \frac{6n+6}{6}\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ &= (n+1)\frac{(n+2)(2n+3)}{6} \end{aligned}$$

   showing it is also true for $n+1$, concluding the proof.
3. Base case: $2{\cdot}3 + 1 = 7 < 8 = 2^3$.

   Induction step: Assume it is true for $k \leq n$ and consider the case $n+1$, we have

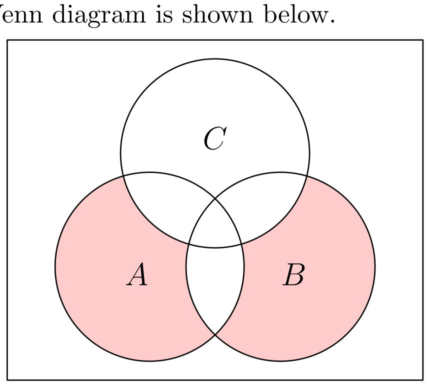
   $$2(n+1)+1 = (2n+1)+2 \leq 2^n+2 \leq 2^n+2^n = 2^{n+1}$$

   showing it is also true for $n+1$, concluding the proof.
4. Done in class.

Sets

1. $x \in \overline{A \cap B}$ if $x \notin A \cap B$. For this to be true we must have $x \notin A$ *or* $x \notin B$, or $x \in \overline{A}$ *or* $x \in \overline{B}$ which shows that $x \in \overline{A} \cup \overline{B}$.

   $x \in \overline{A \cup B}$ if $x \notin A \cup B$. For this to be true we must have $x \notin A$ *and* $x \notin B$, or $x \in \overline{A}$ *and* $x \in \overline{B}$ which shows that $x \in \overline{A} \cap \overline{B}$.
2. Let X be the students taking physics and Y the students taking economics. Then the formula $|X \cup Y| = |X| + |Y| - |X \cap Y|$ gives $(80-13) = 45 + 30 - |X \cap Y|$ or the students taking both are $|X \cap Y| = 8$.
3. The Venn diagram is shown below.



Functions

1. Given $f : X \to Y$ and $g : Y \to Z$, let $z \in Z$. Since g is onto there is some $y \in Y$ so that $g(y) = z$, since f is onto there is some $x \in X$ so that $f(x) = z$. We now have that $(g \circ f)(x) = g(f(x)) = g(y) = z$. This shows that for every z we can find an x that maps to it by $g \circ f$, showing that $g \circ f$ is onto.
2. For functions from X to Y we count the complement. In order to not be an onto function we must have that all elements map to a or all elements map to b. So there are 2 functions which are not onto. On the other hand there are 32 different functions we have that $32 - 2 = 30$ functions are onto.

   Since $|Y| < |X|$ there is no way that a function from Y to X is onto and so in this case we have 0 onto functions.
3. We only need to determine where each element maps. We have:

   $$\begin{aligned} (f \circ f \circ f)(1) &= f(f(f(1))) = f(f(2)) = f(3) = 1 \\ (f \circ f \circ f)(2) &= f(f(f(2))) = f(f(3)) = f(1) = 2 \\ (f \circ f \circ f)(3) &= f(f(f(3))) = f(f(1)) = f(2) = 3 \\ (f \circ f \circ f)(4) &= f(f(f(4))) = f(f(2)) = f(3) = 1 \\ (f \circ f \circ f)(5) &= f(f(f(5))) = f(f(5)) = f(5) = 5 \end{aligned}$$

Sequences

1. We have $q_0 = 0$, $q_1 = 0$, $q_2 = 2$ and so on. Since $n^2 - n$ is increasing for $n \geq 1/2$ (via easy calculus) we see that from 1 on the sequence is increasing, however since $q_0 = q_1$ we have that the sequence is nondecreasing.
2. We have

   $$\begin{aligned} &t_n + 3t_{n-1} \\ &= \left((-3)^n + \frac{1}{2}n + 1\right) + 3\left((-3)^{n-1} + \frac{1}{2}(n-1) + 1\right) \\ &= (-3)^n - (-3)^n + \frac{1}{2}n + \frac{3}{2}n + 1 - \frac{3}{2} + 3 \\ &= 2n + \frac{5}{2}. \end{aligned}$$
3. We have (using the rule for geometric series)

   $$\sum_{k=0}^{n} 2{\cdot}3^k = \frac{2(1-3^{n+1})}{1-3} = 3^{n+1} - 1.$$

   We also have

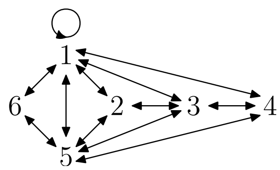
   $$\prod_{k=0}^{n} 2{\cdot}3^k = 2^{n+1}3^{0+1+\cdots+n} = 2^{n+1}3^{n(n+1)/2}.$$

Relations

1. As a set:

$$R = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),$$
$$(2,3),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1),(4,3),$$
$$(4,5),(5,1),(5,2),(5,3),(5,4),(5,6),(6,1),(6,5)\}$$

As a graph:



As a matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Since the only a for which aRa is $a = 1$, we see that this is not reflexive. Since $\gcd(a,b) = \gcd(b,a)$ we have that aRb if and only if bRa so this is symmetric. Since $1R2$ and $2R1$ this is not anti-symmetric. Since $2R5$ and $5R4$ but 2 and 4 are not related this is not transitive.

3. An entry a_{ij} in A^2 is nonnegative if iRp and pRj for some p. For this relationship everything is related to 1 and so this is satisfied for every i and j so all of the entries are nonzero.

Equivalence relations

1. We need to establish the three properties reflexive, symmetric and transitive. Since $a+a = 2a$ is even we have aRa for all a, so it is reflexive. Since $a+b = b+a$ we have that if aRb then bRa, so it is symmetric. If aRb and bRc then $a+b = 2k$ and $b+c = 2\ell$ and so we have

$$a+c = (a+b)+(b+c)-2b = 2k+2\ell-2b = 2(k+\ell-b),$$

showing that aRc, so it is transitive.

There are two equivalence classes $[1] = \{1,3,5\}$, the odd numbers, and $[2] = \{2,4\}$, the even numbers.

2. Since $1+1 = 2$ which is even this relationship cannot be reflexive and so is not an equivalence relationship. Similarly we have $1R2$ and $2R3$ but 1 and 3 are not related, showing that it is not transitive and so again is not an equivalence relationship.

3. Using the rule we have $3!S(6,3) = 540$.

4. b_5 is the sum along the 5th row of the table of Stirling numbers of the second kind, so $b_5 = 52$. Since the number of equivalence relationships is the same as the number of possible partitions there are $b_5 = 52$ different possible equivalence relationships on a set with five elements.

Basic counting

1. By the multiplication rule there are $4{\cdot}5{\cdot}3 = 60$ possible meals.

2. By the multiplication rule there are $6{\cdot}6 = 36$ possible roundtrips. If we insist on taking a different route back then we have $6{\cdot}5 = 30$ possible roundtrips.

3. Each slot has one of 36 possibilities, so by the multiplication principle we have $36{\cdot}36{\cdot}36 = 36^3$ different license plates. If we do not allow repetition of a symbol then there are $36{\cdot}35{\cdot}34 = P(36,3)$.

4. We count the complement. The number of license plates without a number (i.e., only letters) is $26{\cdot}26{\cdot}26 = 26^3$. So the number of license plates with at least one number is $36^3 - 26^3$.

5. We break this into cases. When $k = 0$ there is 1 possible license plate. When $k = 1$ there are 10 possible license plates. When $k = 2$ there are $10{\cdot}10 = 10^2$ possible license plates. ... When $k = 9$ there are $10 \cdots 10 = 10^9$ possible license plates. Now by the addition rule the total number of license plates is

$$1 + 10 + 10^2 + \cdots + 10^9 = 1111111111.$$

More counting

1. Since order matters this is $P(73,6)$.

2. We have to choose four different toppings out of seventeen possibilities which can be done in $\binom{17}{4}$ ways.

3. The number of different seven card hands is the number of ways to pick seven cards out of the 52 which can be done in $\binom{52}{7}$ ways.

4. Counting the number of letters we have 1-M, 4-I, 4-S, 2-P, a total of 11 letters. So there are

$$\frac{11!}{1!\,4!\,4!\,2!}$$

ways to rearrange the letters of the word.

5. Continuing we first arrange the 7 non-S's. This can be done in
$$\frac{7!}{1!\,4!\,2!}$$
ways. We now insert the S's. There are eight slots and the S's have to go into four of the slots. So we can insert the S's in $\binom{8}{4}$ ways so the total number of ways to rearrange without consecutive S's by the multiplication rule is
$$\frac{7!}{1!\,4!\,2!}\binom{8}{4}$$

6. We use bars and stars to distribute the cookies. This can be done in $\binom{9+5-1}{5-1} = \binom{13}{4}$ ways. Similarly we distribute the jelly beans using bars and stars. This can be done in $\binom{34+5-1}{5-1} = \binom{38}{4}$. So by the multiplication rule the number of ways to distribute the goodies is
$$\binom{13}{4}\binom{38}{4}.$$

7. We first give every math major two cookies. There are five cookies left to distribute which can be done in $\binom{5+5-1}{5-1} = \binom{9}{4}$ ways. Similarly we give every computer science major six jelly beans. There are 16 jelly beans left to distribute which can be done in $\binom{16+5-1}{5-1} = \binom{20}{4}$ ways. So by the multiplication rule the number of ways to distribute the goodies is
$$\binom{9}{4}\binom{20}{4}.$$