# Midterm 1 Review

(Test on 20 October 2008)

## Induction

Induction is a method of proof which can often be used for problems that you want to show that something is true for each $n \geq n_0$ (or in other words for $n$ sufficiently large). The key method for every induction problem is to *identify how the current case can be related back to the previous case(s).* Induction proofs are done in two steps

1. Prove the base case(s).
2. Assuming it is true for $k \leq n$ show that it must also be true for $n+1$.

**Examples:**

1. Show that $1+2+\cdots+n = n(n+1)/2$ for $n \geq 1$.
2. Show that $1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$ for $n \geq 1$.
3. Show that $2n+1 \leq 2^n$ for $n \geq 3$.
4. The local store sells rice in 5 lb and 7 lb bags. Show that for every $n \geq 24$ you can always but $n$ pounds of rice.

## Sets

Sets are a collection of objects (called elements). While the objects can be anything we usually will only work with numbers. The order that we list the elements in a set is not important. Let $X, Y$ be sets and $y$ denote some element. Then we have the following notations.

$|X|$ The size of $X$, the number of elements in $X$.

$y \in X$ Way to indicate that $y$ is an element of $X$.

$y \notin X$ Way to indicate that $y$ is *not* an element of $X$.

$X = Y$ Indicates that $X$ and $Y$ are the same set, i.e., they have the same elements.

$X \subseteq Y$ Indicates that $X$ is a subset of $Y$, i.e., every element in $X$ is an element in $Y$.

$\emptyset$ The empty set, a set with no elements.

$X \cup Y$ Denotes union, $\{y \,|\, y \in X \text{ or } y \in Y\}$.

$X \cap Y$ Denotes intersection, $\{y \,|\, y \in X \text{ and } y \in Y\}$.

$\overline{X}$ Denotes complement, $\{y \,|\, y \notin X\}$.

$\mathcal{P}(x)$ Denotes the power set of $X$, the collection of all possible subsets of $X$.

Two sets are disjoint if $X \cap Y = \emptyset$, i.e., they have no common elements. Sets can be represented pictorially using Venn diagrams (useful for understanding statements such as $A \cup (B \cap \overline{C})$). A useful fact is $|X \cup Y| = |X| + |Y| - |X \cap Y|$ (this is a basic example of inclusion-exclusion principle). If $X$ has $n$ elements then $|\mathcal{P}(X)| = 2^n$.

**Examples:**

1. Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
2. There are 80 students. If 45 are taking physics, 30 are taking economics, and 13 are taking neither physics or economics, then how many are taking both?
3. Draw a Venn diagram for sets $A, B, C$. Shade in the region corresponding to $\overline{(B \cap A) \cup C} \cap (A \cup B)$.

## Functions

A function $f : X \to Y$ is a rule which assigns to *every* element in $X$ a *unique* element of $Y$. A function is one-to-one (or injective) if no two elements map to the same value (or in other words each element in $Y$ is hit *at most* once). A function is *onto* (or surjective) if each element of $Y$ is mapped onto by some element in $X$ (or in other words each element in $Y$ is hit *at least* once). A function is bijective if it injective and surjective. If $f : X \to X$ is a bijective function then it is called a permutation since it can be thought of as a rearrangement. There are $n! = n(n-1)(n-2)\cdots 2{\cdot}1$ permutations of $X$ if $|X| = n$. If $f : X \to Y$ and $g : Y \to Z$ then we can "string" the functions together (or compose them) to form a function $g \circ f : X \to Z$ by the rule $(g \circ f)(x) = g(f(x))$.

A useful function $f : \mathbb{R} \to \mathbb{Z}$ is $f(x) = \lfloor x \rfloor$ which maps $x$ to the *largest* integer less than or equal to $x$.

**Examples:**

1. If $f$ is an onto function and $g$ is an onto function show that $g \circ f$ is also onto.
2. Let $X = \{1,2,3,4,5\}$ and $Y = \{a, b\}$. How many onto functions are there from $X$ to $Y$? How many onto functions are there from $Y$ to $X$?
3. Let $X = \{1,2,3,4,5\}$ and let $f : X \to X$ by $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 2$ and $f(5) = 5$. Write down the function for $f \circ f \circ f$.

## Sequences

A sequence is a function from a block of integers (usually of the form $n \geq n_0$) to real or integer values. For example $s(n) = 2^n + 5$, for $n \geq 0$. By convention we can also write $s(n) = s_n$. Note that here $n$ acts as an "indexing" variable and so we can replace it by another symbol (a useful technique!), i.e., $s_i = 2^i + 5$ or $s_{k+2} = 2^{k+2} + 5$ and so on.

Because the integers are ordered we can compare elements. A sequence is *increasing* if $s_n < s_{n+1}$ for each valid $n$, is *nondecreasing* if $s_n \le s_{n+1}$ for each valid $n$, is *decreasing* if $s_n > s_{n+1}$ for each valid $n$, and is *nonincreasing* if $s_n \ge s_{n+1}$ for each valid $n$.

Given a sequence $a_i$ we have

- $\displaystyle\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$.
- A special case of the above is for a geometric sequence, $a(n) = b{\cdot}r^n$ for $n \ge 0$. In this case we have $\displaystyle\sum_{i=m}^{n} b{\cdot}r^i = \frac{b(r^m - r^{n+1})}{1-r}$.
- $\displaystyle\prod_{i=m}^{n} a_i = a_m a_{m+1} \cdots a_n$.

**Examples:**

1. Let $q_n = n^2 - n$ for $n \ge 0$. Is this sequence increasing, nondecreasing, decreasing, nonincreasing or none of the above?
2. Let $t_n = (-3)^n + \frac{1}{2}n + 1$. Simplify the expression $t_n + 3t_{n-1}$.
3. Let $r_k = 2{\cdot}3^k$ for $k \ge 0$. Find simple expressions for $\displaystyle\sum_{k=0}^{n} r_k$ and $\displaystyle\prod_{k=0}^{n} r_k$.

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## Relations

Relations are useful to look at the way that objects are connected to one another. Given sets $X$ and $Y$ the set $X \times Y = \{(x,y) \,|\, x \in X \text{ and } y \in Y\}$. A relation $R$ can be represented as a subset $X \times Y$ and we say $x \in X$ and $y \in Y$ are related, denoted $xRy$, if $(x,y) \in R$. Frequently we will look at *binary* relationships which is when $X = Y$. Relations can also be represented using *digraphs* where each element is a point and for every $(x,y) \in R$ we draw an arrow from $x$ to $y$. Relations can also be represented using matrices. A matrix is an array of numbers $a_{ij}$ where $i$ denotes the row number and $j$ denotes the column number of the location in the matrix. For example,

$$A = i \begin{pmatrix} & \overset{j}{\vdots} & \\ \cdots\cdots & a_{ij} & \cdots \\ & \vdots & \end{pmatrix}$$

Given the relation $R$ we create a 0-1 matrix by

$$a_{ij} = \begin{cases} 1, & \text{if } iRj; \\ 0, & \text{otherwise.} \end{cases}$$

Given matrices $A$ which has size $m \times n$ (i.e., $m$ rows and $n$ columns) and $B$ which has size $n \times p$ then the matrix $C = AB$ has size $m \times p$ and the entries are found by

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j},$$

i.e., multiplying the elements of the $i$th row of $A$ and $j$th column of $B$ and adding up the result.

There are several important properties that (binary) relationships can have (but relationships do not need to have any of these properties).

- *Reflexive* A relationship is reflexive if $xRx$ for each $x \in X$. In terms of the graph this means there is a *loop* at each element; in terms of the matrix this means there are 1s along the diagonal.
- *Symmetric* A relationship is symmetric if when $x \ne y$ and $xRy$ then we also have $yRx$. In terms of the graph this means that whenever we have an edge $x \to y$ we also have the edge $y \to x$; in terms of the matrix this means that $a_{ij} = a_{ji}$, i.e., the matrix is symmetric if we flip along the diagonal.
- *Anti-symmetric* A relationship is anti-symmetric if when $x \ne y$ and $xRy$ then we do not have $yRx$. In terms of the graph this means whenever $x \ne y$ and we have an edge $x \to y$ we do not have the edge $y \to x$; in terms of the matrix this means that if $i \ne j$ and $a_{ij} = 1$ then $a_{ji} = 0$.
- *Transitive* A relationship is transitive if when $xRy$ and $yRz$ then we also have $xRz$. In terms of the graph this means that if we have edges $x \to y$ and $y \to z$ we must also have the edge $x \to z$; in terms of the matrix this means that if $(A^2)_{ij} \le 1$ (the $i,j$th entry of $A^2$) then $a_{ij} = 1$.
- *Partial order* A partial order is a relationship which is reflexive, anti-symmetric and transitive. Examples include "$\le$" on $X = \{1,2,3,\ldots,n\}$ and "$\subseteq$" on $\mathcal{P}(X)$.

**Examples:**

1. The $\gcd(a,b)$ is the biggest number which divides both $a$ and $b$. Let $R$ be the relationship on $X = \{1,2,3,4,5,6\}$ by $aRb$ if $\gcd(a,b) = 1$. Give the three different representations of $R$ (i.e., as a set in $X \times X$, as a graph and as a matrix).
2. What properties does the relationship have in the previous problem (i.e., reflexive, symmetric, anti-symmetric, transitive)? Justify your answer.
3. Given a matrix representation $A$ for the relation in the previous problems, explain why all of the entries of $A^2$ are nonzero.

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## Equivalence relations

An equivalence relation is a relation which is reflexive, symmetric and transitive (i.e., acts like "="). Given an equivalence relationship on a set $X$ we can group the elements into *equivalence classes.* The equivalence class of an element $a$ is $[a] = \{x \,|\, aRx\}$, the idea is to group things which are the "same" under the equivalence relationship.

A *partition* of a set $X$ is a decomposition (or breakdown) of $X$ into nonempty subsets $X_1, X_2, \ldots, X_k$ so that if $i \neq j$ then $X_i \cap X_j = \emptyset$ (i.e., the sets are disjoint) and $X_1 \cup X_2 \cup \cdots \cup X_k = X$ (i.e., the sets cover all of $X$). Given an equivalence relation the equivalence classes give a partition $X$. Similarly given a partition of $X$ we can form an equivalence relationship by saying $aRb$ if $a$ and $b$ are in the same $X_i$. So there is a correspondence between equivalence relationships and partitions.

$S(n,k) = \left\{ {n \atop k} \right\}$, the Stirling numbers of the second kind, count the number of ways to partition an $n$-element set into $k$ nonempty subsets. The number of onto functions from $X$ to $Y$ if $|X| = n$ and $|Y| = k$ is $k!S(n,k)$. $S(n,1) = 1$, $S(n,n) = 1$, $S(n,2) = 2^{n-1} - 1$. Stirling numbers can be found by $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. Table of small valued Stirling numbers:

| $\left\{ {n \atop k} \right\}$ | $k$=1 | $k$=2 | $k$=3 | $k$=4 | $k$=5 | $k$=6 |
|---|---|---|---|---|---|---|
| $n$=1 | 1 | | | | | |
| $n$=2 | 1 | 1 | | | | |
| $n$=3 | 1 | 3 | 1 | | | |
| $n$=4 | 1 | 7 | 6 | 1 | | |
| $n$=5 | 1 | 15 | 25 | 10 | 1 | |
| $n$=6 | 1 | 31 | 90 | 65 | 15 | 1 |

**Examples:**

1. Let $X = \{1,2,3,4,5\}$. Show that the relationship $R$ given by $aRb$ if $a+b$ is even is an equivalence relationship. Find the equivalence classes of this relationship.

2. Let $X = \{1,2,3,4,5\}$. Why is the relationship $R$ given by $aRb$ if $a+b$ is odd not an equivalence relationship?

3. How many onto functions are there from $X = \{1,2,3,4,5,6\}$ to $Y = \{a,b,c\}$?

4. The Bell numbers $b_n$ is a sequence where $b_n$ is the total number of ways to partition an $n$-element set. What is $b_5$? How many different equivalence relationships are possible for a set with five elements?

## Basic counting

- *Multiplication rule* For sets $X_1, X_2, \ldots, X_k$ we have

  $$|X_1 \times X_2 \times \cdots \times X_k| = |X_1||X_2| \cdots |X_k|.$$

  More generally, suppose we have a sequence of choices to be made $B_1, B_2, \ldots, B_k$, where $B_1$ can be done in $b_1$ ways, $B_2$ can be done in $b_2$ ways, and so on. Then the total number of different possibilities is $b_1 b_2 \cdots b_n$. (Note that to use this rule it is important that even though our decision at $B_1$ might change the decision to be made at $B_2$ it does not change the number of ways to make that decision.)

- *Addition rule* For disjoint sets $X_1, X_2, \ldots, X_k$ we have

  $$|X_1 \cup X_2 \cup \cdots \cup X_k| = |X_1| + |X_2| + \cdots + |X_k|.$$

  The idea behind this is "divide and conquer". We can split the problem into several cases, count the number in each situation and then add up the individual cases.

**Examples:**

1. The local diner is running a special where for \$8.95 you can build your own meal where you have a choice of one of four appetizers, one of five main courses and one of three desserts. How many different possible meals are there?

2. You are planning a roundtrip from Los Angeles to New York City and back. There are six possible routes between the two cities. How many different trips are there? What if we insist that we take a different trip route back than we took to get there?

3. The license plates in Mathematicstan consist of a combination of letters ($L = \{a,b,c,\ldots,z\}$) and numbers ($N = \{0,1,\ldots,9\}$). If a license plate has exactly three symbols (each of which can be a letter or a number), how many different license plates are there? What if we do not allow repetition of a symbol?

4. Continuing from the previous problem, what if we insist that a license plate must have *at least* one number (but repetition is OK). Now how many license plates are possible?

5. In the neighboring country of Combinatoria their license plate consists of the ten numbers ($N = \{0,1,\ldots,9\}$) and the first ten letters ($L = \{a,b,\ldots,j\}$) along with the following rule: A license plate must start with a number $k$ and then be followed by $k$ letters (so that the possible different forms are $0, 1L, 2LL, \ldots, 9LLLLLLLLL$). How many different license plates are possible?

## More counting

Selecting $r$ elements out of $n$ objects when *order matters* is called an $r$-permutation of $n$. This can be done in $P(n,r) = n(n-1)\cdots(n-r+1) = n!/(n-r)!$ different ways.

Selecting $r$ elements out of $n$ objects when *order doesn't matter* is called an $r$-combination of $n$. This can be done in

$$C(n,r)=\binom{n}{r}=\frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1}=\frac{n!}{r!(n-r)!}$$

different ways. Table of small valued $\binom{n}{k}$.

| $\binom{n}{k}$ | $k$=0 | $k$=1 | $k$=2 | $k$=3 | $k$=4 | $k$=5 | $k$=6 |
|---|---|---|---|---|---|---|---|
| $n$=0 | 1 | | | | | | |
| $n$=1 | 1 | 1 | | | | | |
| $n$=2 | 1 | 2 | 1 | | | | |
| $n$=3 | 1 | 3 | 3 | 1 | | | |
| $n$=4 | 1 | 4 | 6 | 4 | 1 | | |
| $n$=5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| $n$=6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

The number of ways to arrange $n_1$ objects of type 1, $n_2$ objects of type 2, ..., and $n_k$ objects of type $k$ is

$$\frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\cdots n_k!}.$$

*Bars and Stars.* The number of ways to divide $n$ identical objects into $k$ distinct sets (some of which might be empty) is

$$\binom{n+k-1}{k-1}=\binom{n+k-1}{n}.$$

**Examples:**

1. A magazine editor is laying out a photo spread. She has to put six pictures in out of the 73 that were shot in the studio. How many different ways can she arrange the layout?

2. At Brunette's pizza there are seventeen different toppings. How many different four topping pizzas are available (toppings cannot be repeated)?

3. A normal deck of cards consists of 52 different cards. How many different seven card hands are there?

4. How many ways are there to rearrange the letters of the word "MISSISSIPPI"?

5. How many ways are there to rearrange the letters of the word "MISSISSIPPI" if there cannot be two consecutive S's?

6. There are two math majors and three computer science majors dividing the leftover free food from a talk. They count and discover there are 9 cookies and 34 jelly beans. How many ways are there to divide the cookies and jelly beans among the five students?

7. Continuing the previous problem, after some tense negotiation they agree that every math major will get at least two cookies and every computer science major will get at least six jelly beans. Now how many ways are there to divide the cookies and jelly beans?