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MATH 61 (Butler)
Final, 12 December 2008

This test is closed book and closed notes, with the exception that you are allowed one $8\frac{1}{2}'' \times 11''$ page of handwritten notes. For full credit show all of your work (legibly!), unless otherwise specified. You do not need to simplify terms such as “5!” or “ $\binom{13}{5}$ ”. Each problem is worth 10 points.

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1. Show that if a tree T has a vertex of degree m then it has *at least* m vertices of degree one (i.e., m leaves).

Method the first:

We know that there are $n - 1$ edges and so the sum of the degrees is $2n - 2$ (i.e., twice the number of edges. Suppose that we have k leaves, then we have a vertex of degree m and so all the remaining $n - k - 1$ vertices have degree at least 2 and so we have

$$2n - 2 = \sum_v d(v) \geq m + k + 2(n - k - 1).$$

If we now cancel terms and rearrange this gives us

$$k \geq m,$$

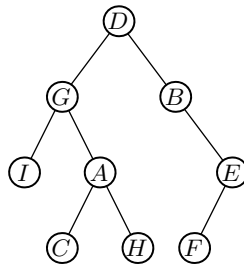
showing that we have at least m leaves.

Method the second:

Starting at the vertex of degree m we can form m simple paths so that each path going along a different edge incident to m and each one terminating at a leaf. If any two of the leaves that we ended up at were the same we could combine the two simple paths and with some trimming get a simple cycle, but trees are acyclic. Therefore each of these m leaves are distinct and so we have at least m leaves.

2. Draw a labeled binary tree where the *preorder* of the vertices is $DGIACHBEF$ and the *inorder* of the vertices are $IGCAHDBFE$ (make sure the vertices are labeled and left/right is clearly distinguished).

Before we begin let us outline our general technique. In the preorder the first item listed is the root (D), if we now use this we can look at the inorder and find the location of the root in the listing. We can now split the inorder in half the first half ($IGCAH$) being what goes to the left tree, the second half (BFE) being what goes to the right tree. For each one of these trees we have the inorder but we also can recover the preorder, for the left it will be $GIACH$ and for the right it will be BEF . And now we can continue this process until we have identified the whole tree. In our case the final answer is shown below.



3. Fullerenes are molecules composed entirely of carbon, the most famous of which is Buckminsterfullerene (C_{60}) which is composed of 60 carbon atoms in the shape of a soccer ball. These three dimensional molecules can be represented in the plane by connected planar graphs where each vertex corresponds to a carbon atom and edges represent bonds between atoms. In the graph corresponding to the fullerene each vertex has degree three and each face must be either a pentagon (five-sided) or a hexagon (six-sided).

Show that in every fullerene there are *exactly* twelve pentagons.

Before we begin let us observe some key words mentioned in the problem, namely, “connected”, “planar graphs”, “faces”. Looking back there is only one idea that we studied which has all of these ingredients: Euler’s formula,

$$V - E + F = 2.$$

So let us try to use that to answer our question.

Since every face is either a pentagon or a hexagon let us denote the number of pentagons by \diamond and the number of hexagons by \square . Then clearly we have

$$F = \diamond + \square.$$

Also we can count the number of edges by adding up the number of edges used in each face, since each edge will show up in two faces we also have

$$2E = 5\diamond + 6\square \quad \text{or} \quad E = \frac{5}{2}\diamond + 3\square.$$

Finally, since the graph is cubic, i.e., each vertex has degree 3 and so we have that $3V = 2E$ or $V = \frac{2}{3}E$.

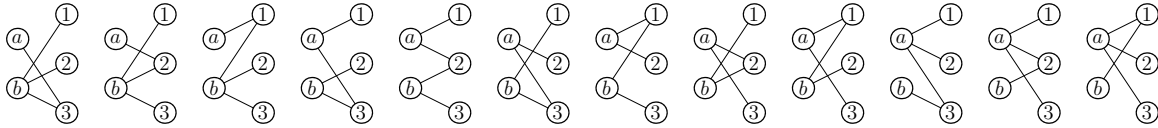
Putting all of this together we have

$$2 = V - E + F = -\frac{1}{3}E + F = -\frac{1}{3}\left(\frac{5}{2}\diamond + 3\square\right) + (\diamond + \square) = \frac{1}{6}\diamond.$$

Or simplifying we have $\diamond = 12$, showing that there are exactly twelve pentagons.

4. We saw that the number of labeled trees is n^{n-2} . Alternatively this says that the number of different spanning trees of K_n is n^{n-2} . Find the number of different spanning trees of $K_{2,n}$ (with vertices of the two parts labeled $\{a, b\}$ and $\{1, 2, \dots, n\}$ respectively). For example there are 12 different spanning trees for $K_{2,3}$ and these are shown below.

Justify your answer.



Since the graph is connected there has to be a path connecting a and b . In particular there has to be some vertex in $\{1, 2, \dots, n\}$ that is adjacent to both a and b at the same time. On the other hand if there were two vertices adjacent to both a and b at the same time we could form a simple cycle of length 4, and so we would not have a tree. Also note that every vertex in $\{1, 2, \dots, n\}$ has to link to either a and b (otherwise the graph is not connected). In particular we have the following:

- There is exactly one vertex in $\{1, 2, \dots, n\}$ adjacent to both a and b .
- Every other vertex is adjacent to exactly one of a and b .

It is easy to see that a graph satisfying both of these is connected and has $n + 1$ edges and so it must be a tree. So to form a spanning tree we need to specify which vertex is adjacent to both a and b , this can be done in n ways. For the remaining $n - 1$ vertices we have to decide to connect to either a or b , so we have 2 choices for each of the remaining $n - 1$ terms. So by the multiplicative rule we have that the number of spanning trees is

$$n2^{n-1}.$$

(Note: that for $n = 1$ this gives us 1, for $n = 2$ this gives us 4 and for $n = 3$ this gives us 12 which agrees with the picture above.)

5. Recall the the Fibonacci numbers satisfy $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Use induction to show that for $n \geq 2$

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n.$$

First we need to establish the base case:

For $n = 2$ we have $F_3F_1 = 2$ and $F_1 + (-1)^2 = 2$ and so it holds for $n = 2$. Now assume that it holds for $k \leq n$ and let us consider the case $k = n + 1$:

First note that we can rearrange the recursion for the Fibonacci numbers as follows,

$$F_{n-2} = F_n - F_{n-1} \quad \text{or shifting} \quad F_{n-1} = F_{n+1} - F_n.$$

By the induction hypothesis we have that

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n.$$

If we now substitute for F_{n-1} using the above we have

$$F_{n+1}(F_{n+1} - F_n) = F_n^2 + (-1)^n,$$

or if we expand and rearrange

$$F_{n+1}^2 - (-1)^n = F_n^2 + F_nF_{n+1} = F_n(F_n + F_{n+1}).$$

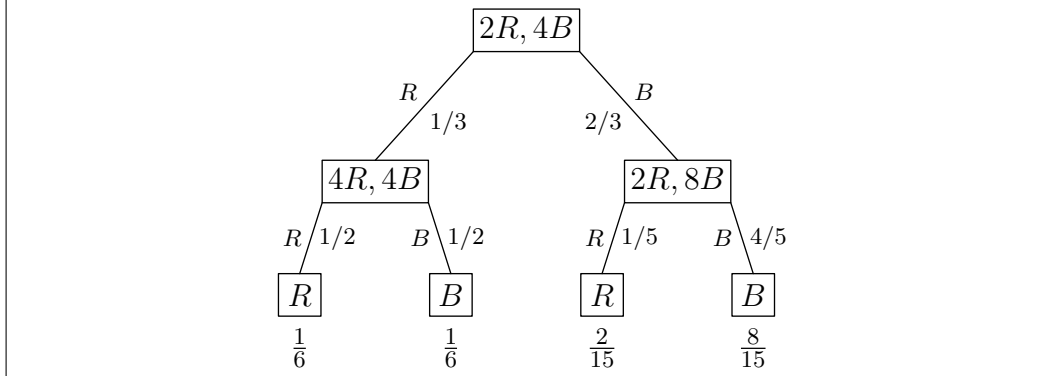
Finally, we can move the negative into the $(-1)^n$ term and use the recursion for the Fibonacci numbers for what is on the right to get

$$F_{n+1}^2 + (-1)^{n+1} = F_nF_{n+2},$$

which establishes the next case and concludes the proof by induction.

6. There is an urn which initially has 2 red and 4 blue marbles. You first draw out a marble. If it is red, put the marble back and add 2 additional red marbles. If it is blue, put the marble back and add 4 additional blue marbles. You now draw out a second marble.

Before we attempt to answer this question, let us draw a tree that represents what is going on in this problem.



- (a) What is the probability that the second marble is red?

Now with the above tree this is the same as adding up the probabilities corresponding to the two leaves where we draw an R in the second round. In particular we have that the probability the second marble is red is

$$\frac{1}{6} + \frac{2}{15} = \frac{9}{30}.$$

- (b) What is the probability that the first marble you drew was blue, given that the second marble is red?

By conditional probability and using the tree and part (a) this is

$$\frac{P(\text{first drew blue AND second drew red})}{P(\text{second drew red})} = \frac{2/15}{9/30} = \frac{4}{9}.$$

7. A bin of light bulbs has five bad light bulbs and twelve good light bulbs. If you take out six light bulbs at random what is the probability that *at most* two of them are defective?

This is really a counting problem. There are a total of 17 light bulbs and so the total number of ways for us to pick out 6 of them is $\binom{17}{6}$. We also need to figure out how many ways we can pick light bulbs out so that there are at most two which are defective. To do this we can break it into cases and use the addition rule. Namely we count the number of ways where we choose no defective light bulbs and six good light bulbs ($\binom{5}{0}\binom{12}{6}$), the number of ways where we choose one defective light bulb and five good light bulbs ($\binom{5}{1}\binom{12}{5}$) and the number of ways where we choose two defective light bulbs and four good light bulbs ($\binom{5}{2}\binom{12}{4}$).

Putting this altogether we have that the probability that at most two are defective is

$$\frac{\binom{5}{0}\binom{12}{6} + \binom{5}{1}\binom{12}{5} + \binom{5}{2}\binom{12}{4}}{\binom{17}{6}} = \frac{4917}{6188}.$$

(Note you were not required to compute the fraction, that has been added merely for your enjoyment.)

8. A group of four students ordered a super-large pizza which comes with 32 slices of gooey goodness for their late night study session. How many ways are there to divide up the pizza if every student gets at least two slices?

This is a bars and stars problem. First off we can distribute two slices to each of the four people. That leaves us with 24 slices which we can now distribute arbitrarily among the four people. This can be done in

$$\binom{24 + 4 - 1}{4 - 1} = \binom{27}{3} = 2925 \text{ ways.}$$

(Again you did not have to compute the 2925, this was added merely for your enjoyment.)

9. Give two proofs for the following identity

$$n2^{n-1} = \sum_{k=0}^n k \binom{n}{k}.$$

(a) By using the binomial theorem. (Hint: start with $(1+x)^n$ and take derivatives.)

By the binomial theorem we have

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

Taking the derivative of each side with respect to x we have

$$n(1+x)^{n-1} = \sum_{k=0}^n k \binom{n}{k} x^{k-1}.$$

This is true for all values of x and so now substituting in the value $x = 1$ we are left with

$$n2^{n-1} = \sum_{k=0}^n k \binom{n}{k}.$$

(b) By giving a combinatorial proof. Namely, count the number of ways to form a group with a leader out of a pool of n people.

There are two ways to form a group with a leader out of a pool of n people.

Method 1: First we pick a leader (which can be done in n different ways), then for each of the remaining $n - 1$ people they are either in or out of the group, so for the remaining $n - 1$ people there are 2 possibilities for each of them. So by the multiplication rule the total number of ways using this method is

$$n2^{n-1}.$$

Method 2: For each $k = 0, 1, \dots, n$ we first pick a group of size k (which can be done in $\binom{n}{k}$ ways). Then out of that group of k people we select one of them to be the leader (which can be done in k ways). So the total number of groups of size k with a leader is $k \binom{n}{k}$. Now using the addition rule we have that the total number of ways using this method is

$$\sum_{k=0}^n k \binom{n}{k}.$$

Since we will get the same number regardless of the method we can conclude that

$$n2^{n-1} = \sum_{k=0}^n k \binom{n}{k}.$$

10. For a fixed value of r consider the sequence defined by

$$g_n = \sum_k r^k \binom{n-k}{k}.$$

(a) Show that $g_0 = 1$, $g_1 = 1$ and $g_{n+1} = g_n + rg_{n-1}$ for $n \geq 1$.

Before we begin a quick note about notation, “ \sum_k ” means that we add up over all values of k , in our case only finitely many of these values are finite and so we could rewrite it as a finite sum. However, in problems like this it is sometimes easier to suppress our exact range for the value of k , we could add it, but it would only make what we are about to do notationally messy.

We have

$$g_0 = \sum_k r^k \binom{0-k}{k} = r^0 \binom{0}{0} = 1,$$

and

$$g_1 = \sum_k r^k \binom{1-k}{k} = r^0 \binom{1}{0} = 1.$$

For the third part we have the following.

$$\begin{aligned} g_n + rg_{n-1} &= \sum_k r^k \binom{n-k}{k} + r \sum_k r^k \binom{n-1-k}{k} \\ &= \sum_k r^k \binom{n-k}{k} + \sum_k r^{k+1} \binom{n-1-k}{k} \\ &= \sum_\ell r^\ell \binom{n-\ell}{\ell} + \sum_\ell r^\ell \binom{n-\ell}{\ell-1} \\ &= \sum_\ell r^\ell \left(\binom{n-\ell}{\ell} + \binom{n-\ell}{\ell-1} \right) \\ &= \sum_\ell r^\ell \binom{n+1-\ell}{\ell} \\ &= g_{n+1} \end{aligned}$$

The first line is using the definition of g_n , the second we move the r term inside, on the third we reindex our sum (this is the hardest step!) on the fourth we collect, on the fifth we used the basic binomial identity $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ and finally we again used the definition of g_n .

(b) Solve the recurrence for g_n in the special case $r = -1/4$.

We have $g_{n+1} = g_n - \frac{1}{4}g_{n-1}$ with $g_0 = g_1 = 1$. This is a linear homogeneous recursion with constant coefficients of order 2 the kind that we like). So to solve we translate this into a polynomial problem. Namely this becomes

$$t^2 = t - \frac{1}{4} \quad \text{or} \quad 0 = t^2 - t + \frac{1}{4} = \left(t - \frac{1}{2}\right)^2.$$

So we have a double root of $\frac{1}{2}$. Therefore our solution will take the form

$$g_n = A\left(\frac{1}{2}\right)^n + Bn\left(\frac{1}{2}\right)^n,$$

where A and B are constants that will be determined using the initial conditions. We have

$$1 = g_0 = A \quad \text{and} \quad 1 = g_1 = \frac{1}{2}A + \frac{1}{2}B.$$

From these equations it is easy to solve and see that $A = B = 1$ and so the desired solution is

$$g_n = \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n = \frac{n+1}{2^n}.$$