Final Review Solutions

1. For $n = 1$ a tournament is a single vertex which is a Hamiltonian path. For $n = 2$ a tournament is of the form $a \to b$ which again is a Hamiltonian path. So the base case is established.

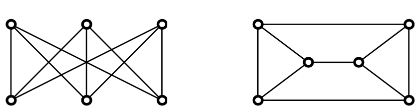
   Suppose that we know it is true up through k vertices. Let us consider the case of $k + 1$ vertices. First we note that we can take k of the vertices and these vertices form a tournament and so by our induction hypothesis we can label them so that there is a Hamiltonian path of the form $1 \to 2 \to \cdots \to k$. Now let us consider the vertex $k + 1$. If the arc between $k + 1$ and 1 is $(k + 1) \to 1$ then we can modify the path as $(k + 1) \to 1 \to \cdots k$ and we are done. Similarly, if the arc between $k+1$ and k is $k \to (k+1)$ then we can modify the path as $1 \to \cdots \to k \to (k + 1)$ and we are done. So now suppose that we have arcs $1 \to k$ and $(k + 1) \to k$, then there is some smallest $1 < i \le k$ so that $(k + 1) \to i$, note that we must have $(i - 1) \to (k + 1)$. Using this we can form the path $1 \to \cdots (i - 1) \to (k+1) \to i \to \cdots \to k$ giving us the desired Hamiltonian path and concluding the proof.

2. The entries of A^2 count the number of walks of length 2 joining vertices. The number of walks of length 2 is also the same as the number of common vertices shared. So we have that

$$(A^2)_{i,j} = \begin{cases} k & \text{if } i = j; \\ \mu & \text{if } i \text{ adjacent to } j; \\ \nu & \text{otherwise.} \end{cases}$$

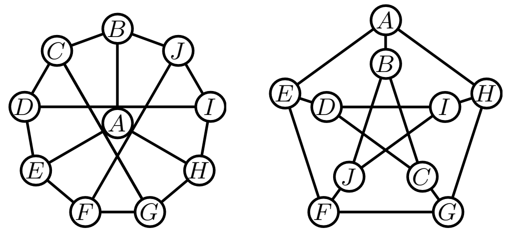
3. We need to check that the graph is regular, which it is showing that $k = 3$. That between adjacent vertices they share the same number of common neighbors, but since there are no triangles then adjacent vertices have no common neighbors and so $\mu = 0$. Finally between nonadjacent vertices they share the same number of common neighbors, this can be checked individually but by using the symmetry of the graph we only need check one vertex and it is easy to see that $\nu = 1$.

4. Consider the following two graphs.



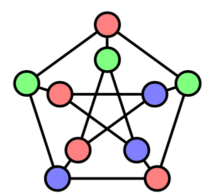
   Both have six vertices and are regular of degree three. But they are not isomorphic, for example the graph on the right has a cycle of length three but the graph on the left does not.

5. Below are labelings on the two graphs. It is now easy to check that they have the same adjacency relationships and so the graphs are isomorphic.



6. True. We need to show that if G and H are isomorphic graphs that they have the same chromatic number. Given a coloring of G (an assignment of colors to the vertices so that no two vertices have the same color) then using the bijective map $f : V(G) \to V(H)$ we assign the vertex $f(v)$ the same color as v. This gives a coloring on H with no two vertices the same color. In particular this shows that $\chi(H) \le \chi(G)$. But by a similar argument we have that $\chi(G) \le \chi(H)$ and so we can conclude that $\chi(G) = \chi(H)$ and so the chromatic number is an invariant.

7. Clearly one color is not enough, and since the graph has a closed cycle of length five the graph is not bipartite and so two colors are not enough. However we can color the Petersen graph with three colors (see below) and so it has chromatic number three.



8. (We also need the assumption of connected.) Each face has three edges and each edge is used in two faces. So combining this we have $3F = 2E$, or $E = 3/2F$. Now using Euler's formula we have

$$2 = V - E + F = V - \frac{3}{2}F + F = V - \frac{1}{2}F,$$

   which we can rearrange to give $F = 2V - 4$.

9. (We also need the assumption of connected.) Each vertex is incident to three edges and each edge is incident to two vertices. So combining this we have $3V = 2E$, or $E = 3/2V$. Now using Euler's formula we have

$$2 = V - E + F = V - \frac{3}{2}V + F = -\frac{1}{2}V + F,$$

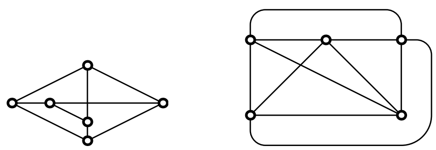
   which we can rearrange to give $V = 2F - 4$.

10. If G and H are dual graphs then every vertex of G is a face of H, and every face of G is a vertex of H. So $V_G = F_H$ and $F_G = V_H$. Also it is easy to see that the dual of a connected planar graph is another connected planar graph and so we have

$$V_G - E_G + F_G = 2 = V_H - E_H + F_H.$$

Canceling terms this reduces to $E_G = E_H$ showing that they have the same number of edges.

11. Below are drawings for $K_{3,3}$ and K_5 with only one crossing each.



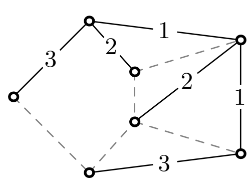
12. The height of a tree is the longest simple path with the starting vertex at the root. In particular the height of any tree is bounded by the length of the longest path and so $h \le \kappa$.

    On the other hand, in any rooted tree the longest possible simple path has length at most $2h$. To see this we can cut the path into two pieces, on the first piece we move up levels and on the second we move down. On each part we can use at most h edges and so the simple path has at most $2h$ edges. In particular we see that $\kappa \le 2h$ or $h \ge \kappa/2$, but since h is an integer we can round the term on the right up and can conclude $h \ge \lceil \kappa/2 \rceil$.

13. The relationship is anti-symmetric and transitive.

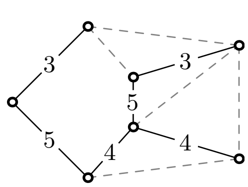
14. To form a spanning tree we have to remove exactly one edge from each triangle. Since each triangle has three edges this can be done in $3{\cdot}3{\cdot}3 = 27$ ways, so there are 27 spanning trees.

15. Below is the minimal spanning tree (in this case there is only one). The sum of the edge weights is 12.



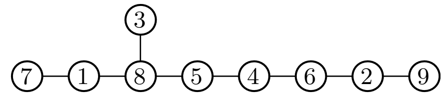
16. One possible algorithm is as follows: Starting with the whole graph, remove the edge with the *smallest* weight which does not disconnect the graph. Repeat until no more edges can be removed, the resulting graph is the desired tree.

    Below is the maximal spanning tree (again in this case there is only one). The sum of the edge weights is 24.

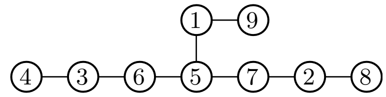


17. Using the technique given in the review the Prüfer code is 36361319.

18. (a) For 81854629 the tree is the following.



    (b) For 36527519 the tree is the following.

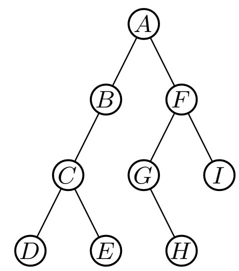


19. The preorder is $DBFICAHGE$.

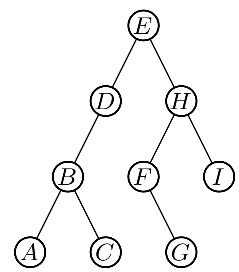
    The inorder is $IFCBDHGAE$.

    The postorder is $ICFBGHEAD$.

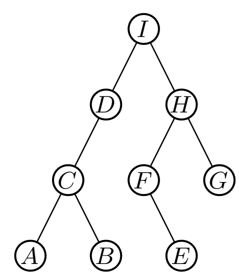
20. The labeling is below.



21. For the inorder we get the following labeling.



    While for the postorder we get the following labeling.



22. The reverse-inorder is $EAGHDBCFI$.

    In general the reverse-inorder is found by taking the reverse of the inorder (as the name highly suggests).

23. Given the preorder and inorder we can construct the labeled tree that this corresponds to. In our case it is not too hard to see that starting at the root every vertex has a single *right* child, and then these vertices are labeled (starting at the root) $A, B, \ldots, N$. Using this we have that the postorder is $NMLKJIHGFEDCBA$.

24. The probability that at most two of the coins are heads is the sum of the probabilities that none of the coins are heads, one is a head and two are heads. In our case since there are 2^7 possible outcomes for the flips we have that the probability is

$$\frac{\binom{7}{0}+\binom{7}{1}+\binom{7}{2}}{2^7}=\frac{29}{128}.$$

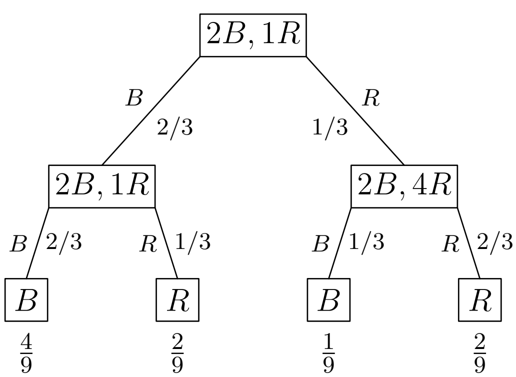
25. Using bars and start we first note that there are $\binom{20+5-1}{5-1}$ ways to divide the cookies among the four people. On the other hand the number of ways to divide the cookies up so that everyone gets at least two is to first give each person two cookies then divide up the remaining ten cookies which can be done in $\binom{10+5-1}{5-1}$ ways. So the probability is

$$\frac{\binom{14}{4}}{\binom{24}{4}}=\frac{13}{138}.$$

26. The total number of ways (with respect to gender) that the seating arrangement can be made without restriction is $\binom{7}{3}$ (i.e., choose three out of the seven chairs for the girls to sit in). The number of ways (with respect to gender) that the seating arrangement can be made so that no two girls sit next to each other is $\binom{5}{3}$ (i.e., line up the four boys, there are now five gaps that girls can go into, i.e., $\_B\_B\_B\_B\_$ and at most girl can go into each gap so we choose three out of these five to be filled). Therefore the probability that no two girls sit next to each other is

$$\frac{\binom{5}{3}}{\binom{7}{3}}=\frac{2}{7}.$$

27. Let us first draw a tree representing what is going on.



On the edges we indicate the result of the current step plus the corresponding probability. At the very bottom we have listed the probability of getting to that leaf. With this tree in hand it is now easy to answer the questions by using the appropriate leaves. We will use $P(1B|2R)$ indicate the probability that the first was red given the second was blue, and other terms will be similarly defined.

(a) $P(1R|2R)=\dfrac{P(1R\cap 2R)}{P(2R)}=\dfrac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{9}}=\dfrac{1}{2}.$

(b) $P(1R|2B)=\dfrac{P(1R\cap 2B)}{P(2B)}=\dfrac{\frac{1}{9}}{\frac{4}{9}+\frac{1}{9}}=\dfrac{1}{5}.$

(c) $P(1B|2R)=\dfrac{P(1B\cap 2R)}{P(2R)}=\dfrac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{9}}=\dfrac{1}{2}.$

(d) $P(1B|2B)=\dfrac{P(1B\cap 2B)}{P(2B)}=\dfrac{\frac{4}{9}}{\frac{4}{9}+\frac{1}{9}}=\dfrac{4}{5}.$

28. (a) To calculate the probability that we flipped exactly two heads we can "carve" up our probability space depending on what we rolled on the die. Notationally, let $P(2H|R3)$ denote the probability that we flipped 2 heads given that we rolled a 3. Then we have that the probability we flipped 2 heads is

$$P(2H|R1)P(R1)+\cdots+P(2H|R6)P(R6)$$
$$=\frac{\binom{1}{2}}{2^1}\frac{1}{6}+\cdots+\frac{\binom{6}{2}}{2^6}\frac{1}{6}=\frac{33}{128}.$$

(b) We have that

$$P(R4|2H)=\frac{P(R4\cap 2H)}{P(2H)}=\frac{\frac{1}{6}\cdot\frac{\binom{4}{2}}{2^4}}{\frac{33}{128}}=\frac{8}{33}.$$

29. Again let us carve up our probability space depending on whether or not we have senioritus (let S indicate have senioritus and $\overline{S}$ indicate not have senioritus, similarly let T indicate test positive and $\overline{T}$ indicate test negative). We now have

$$P(T)=P(T|S)P(S)+P(T|\overline{S})P(\overline{S})$$
$$=1\cdot\frac{1}{50}+\frac{1}{20}\cdot\frac{49}{50}=\frac{69}{1000}.$$

Using this we have

$$P(S|T)=\frac{P(T|S)P(S)}{P(T)}=\frac{\frac{1}{50}}{\frac{69}{1000}}=\frac{20}{69}.$$

In particular, even if you test positive for senioritus it is not likely that you have the disease, you might just be a natural procrastinator.

30. (a) Since the flips are independent then the probability it comes up HHT is $p\cdot p\cdot(1-p)$, similarly for HTH and THH. So the probability is

$$3p^2(1-p).$$

(b) Using calculus to maximize the above expression (treating p as a variable), we see that the maximum will occur when

$$0 = \frac{d}{dp}(3p^2 - 3p^3) = 6p - 9p^2 = 3p(2 - 3p).$$

This gives us the possibilities of $p = 0$ and $p = 2/3$. Clearly $p = 0$ corresponds to a minimum so the correct value is $p = 2/3$.

31. One way to test if two events A and B are independent we compute $P(A)$, $P(B)$ and $P(A \cap B)$ and check whether $P(A \cap B) = P(A)P(B)$.

(a) Let A be the event the first die is a 2 and let B be the even that the sum of the two dies is 7. Since the die is fair we have that $P(A) = 1/6$, $P(B) = 1/6$ (since there are 6 out of 36 ways that the two die can sum to seven) while $P(A \cap B) = 1/36$ (since the only way both events can happen is for the first die to be a 2 and the second to be a 5). Since $P(A \cap B) = P(A)P(B)$ these two events are independent.

(b) Let A be the event the first die is a 2 and let B be the even that the sum of the two dies is 8. Since the die is fair we have that $P(A) = 1/6$, $P(B) = 5/36$ (since there are 5 out of 36 ways that the two die can sum to seven) while $P(A \cap B) = 1/36$ (since the only way both events can happen is for the first die to be a 2 and the second to be a 6). Since $P(A \cap B) \neq P(A)P(B)$ these two events are *not* independent.