Topics in Discrete Mathematics

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Assignment 3

Due: December 28

Solution of every problem should be no longer than one page!

Problem 1: Let $A_1, \ldots, A_m$ be subsets of an n-element set such that $|A_i|$ is not divisible by 6 for every i, but the sizes of all pairwise intersections $|A_i \cap A_j|$ are divisible by 6. Prove that $m \leq 2n$.

Problem 2: Let $A_1, \ldots, A_m$ be subsets of an n-element set. Assume that their pairwise symmetric differences $A_i \triangle A_j = (A_i \setminus A_j) \cup (A_j \setminus A_i)$ have only two sizes. Prove that $m \leq \frac{n(n+1)}{2} + 1$. Find $m = \frac{n(n-1)}{2} + 1$ subsets of an n-element set with only two sizes of symmetric differences.

Problem 3: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an n-element set with the property that $|A \cap B|$ is *odd* for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that then $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

Problem 4: Prove that a connected graph G with maximum eigenvalue λ_1 is bipartite if and only if $-\lambda_1$ is also an eigenvalue.

Problem 5: Let A_G be the adjacency matrix of graph G which has only k distinct eigenvalues. Show that the diameter of G is greater than k.

Problem 6: Let λ_1 be the maximum eigenvalue of graph G. Prove that the chromatic number of G is at most $\lambda_1 + 1$.