Getting a directed Hamilton cycle two times faster

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Abstract

Consider the random graph process where we start with an empty graph on n vertices, and at time t, are given an edge e_t chosen uniformly at random among the edges which have not appeared so far. A classical result in random graph theory asserts that *whp* the graph becomes Hamiltonian at time $(1/2+o(1))n\log n$. On the contrary, if all the edges were directed randomly, then the graph has a directed Hamilton cycle *whp* only at time $(1+o(1))n\log n$. In this paper we further study the directed case, and ask whether it is essential to have twice as many edges compared to the undirected case. More precisely, we ask if at time t, instead of a random direction one is allowed to choose the orientation of e_t, then whether it is possible or not to make the resulting directed graph Hamiltonian at time earlier than $n\log n$. The main result of our paper answers this question in the strongest possible way, by asserting that one can orient the edges on-line so that *whp*, the resulting graph has a directed Hamilton cycle exactly at the time at which the underlying graph is Hamiltonian.

1 Introduction

The celebrated *random graph process*, introduced by Erdős and Rényi [11] in the 1960's, begins with an empty graph on n vertices, and in every round $t=1,\dots,m$ adds to the current graph a single new edge chosen uniformly at random out of all missing edges. This distribution is commonly denoted as $G_{n,m}$. An equivalent "static" way of defining $G_{n,m}$ would be: choose m edges uniformly at random out of all $\binom{n}{2}$ possible ones. One advantage in studying the random graph process, rather than the static model, is that it allows for a higher resolution analysis of the appearance of monotone graph properties (a graph property is monotone if it is closed under edge addition).

A *Hamilton cycle* of a graph is a simple cycle that passes through every vertex of the graph, and a graph containing a Hamilton cycle is called *Hamiltonian*. Hamiltonicity is one of the most fundamental notions in graph theory, and has been intensively studied in various contexts, including random graphs. The earlier results on Hamiltonicity of random graphs were obtained by Pósa [20], and Korshunov [17]. Improving on these results, Komlós and Szemerédi [16] proved that if $m'=\frac{1}{2}n\log n+\frac{1}{2}\log\log n+c_n n$, then

$$\lim_{n\to\infty}\mathbb{P}(G_{n,m'}\text{ is Hamiltonian})=\begin{cases}0 & \text{if } c_n\to-\infty\\ e^{-e^{-2c}} & \text{if } c_n\to c\\ 1 & \text{if } c_n\to\infty.\end{cases}$$

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One obvious necessary condition for the graph to be Hamiltonian is for the minimum degree to be at least 2, and surprisingly, the probability of $G_{n,m'}$ having minimum degree two at time m' has the same asymptotic behavior as the probability of it being Hamiltonian. Bollobás [7] strengthened this observation by proving that *whp* the random graph process becomes Hamiltonian when the last vertex of degree one disappears. Moreover, Bollobás, Fenner, and Frieze [8] described a polynomial time algorithm which *whp* finds a Hamilton cycle in random graphs.

Hamiltonicity has been studied for directed graphs as well. Consider a *random directed graph process* where at time t a random directed edge is chosen uniformly at random among all missing edges. and let $D_{n,m}$ be the graph consisting of the first m edges. Frieze [14] proved that for $m'' = n \log n + c_n n$, the probability of $D_{n,m''}$ containing a (directed) Hamilton cycle is

$$\lim_{n\to\infty} \mathbb{P}(D_{n,m''} \text{ is Hamiltonian}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-2e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Similar to the undirected case, this probability has the same asymptotic behavior as the probability of the directed graph having minimum in-degree and out-degree 1. In fact, Frieze proved [14] that when the last vertex to have in- or out-degree less than one disappears, the graph has a Hamilton cycle *whp*.

Hamiltonicity of various other random graph models has also been studied [21, 3]. One model which will be of particular interest to us is the k-in k-out model, in which every vertex chooses k in-neighbors and k-out neighbors uniformly at random and independently of the others. Improving on several previous results, Cooper and Frieze [9] proved that a random graph in this model is Hamiltonian *whp* already when $k = 2$ (which is best possible since it is easy to see that a 1-in 1-out random graph is *whp* not Hamiltonian).

1.1 Our Contribution

Bollobás [7], and Frieze's [14] results introduced above suggest that the main obstacle to Hamiltonicity of random graphs lies in "reaching" certain minimum degree conditions. It is therefore natural to ask how the thresholds change if we modify the random graph process so that we can somehow bypass this obstacle.

We consider the following process suggested by Frieze [15] which has been designed for this purpose. Starting from the empty graph, at time t, an undirected edge (u, v) is given uniformly at random out of all missing edges, and a choice of its orientation ($u \to v$ or $v \to u$) is to be made at the time of its arrival. In this process, one can attempt to accelerate the appearance of monotone directed graph properties, or delay them, by applying an appropriate on-line algorithm. It is important to stress that the process is *on-line* in nature, namely, one cannot see any future edges at the current round and is forced to make the choice based only on the edges seen so far. In this paper, we investigate the property of containing a directed Hamilton cycle by asking the question, "can one speed up the appearance of a directed Hamilton cycle?". The best we can hope for is to obtain a directed Hamilton cycle at the time when the underlying graph has minimum degree 2. The following result asserts that directed Hamiltonicity is in fact achievable exactly at that time, and this answers the above question positively in the strongest possible way.

Theorem 1.1. *Let $\mathcal{G}$ be a random (undirected) graph process that terminates when the last vertex of degree one disappears. There exists an on-line algorithm* **Orient** *that orients the edges of $\mathcal{G}$, so that the resulting directed graph is Hamiltonian whp.*

Let us remark that $\mathcal{G}$ *whp* contains $(1 + o(1))n \log n/2$ edges, in contrast with $(1 + o(1))n \log n$ edges in the random directed graph model. Thus the required number of random edges is reduced by half.

Our model is similar in spirit to the so called Achlioptas process. It is well known that a giant connected component (i.e. a component of linear size) appears in the random graph $G_{n,m}$ when $m = (1 + o(1))n/2$. Inspired by the celebrated "power of two choices" result [2], Achlioptas posed the following question: Suppose that edges arrive in pairs, that is in round t the pair of edges (e_t, e'_t) chosen uniformly at random is given, and one is allowed to pick an edge out of it for the graph (the other edge will be discarded). Can one delay the appearance of the giant component? Bohman and Frieze answered this question positively [4] by describing an algorithm whose choice rule allows for the ratio $m/n \geq 0.53$, and this ratio has been improved since [5]. Quite a few papers have thereafter studied various related problems that arise in the above model [6, 13, 18, 22, 23]. As an example, in [18], the authors studied the question, "How long can one delay the appearance of a certain fixed subgraph?".

One such paper which is closely related to our work is the recent work of Krivelevich, Lubetzky, and Sudakov [19]. They studied the Achlioptas process for Hamiltonicity, and proved that by exploiting the "power of two choices", one can construct a Hamilton cycle at time $(1 + o(1))n \log n/4$, which is twice as fast as in the random case. Both our result and this result suggest that the "bottleneck" to Hamiltonicity of random graphs indeed lies in the minimum degree, and thus these results can be understood in the context of complementing the results of Bollobás [7], and Frieze [14].

1.2 Preliminaries

The paper is rather involved technically. One factor that contributes to this is the fact that we are establishing the "hitting time" version of the problem. That is, we determine the exact threshold for the appearance of a Hamilton cycle. The analysis can be simplified if one only wishes to estimate this threshold asymptotically (see concluding remarks). To make the current analysis more approachable without risking any significant change to the random model, we consider the following variant of the graph process, which we call the *random edge process* : at time t, an edge is given as an ordered pair of vertices $e_t = (v_t, w_t)$ chosen uniformly at random, with repetition, from the set of all possible n^2 ordered pairs (note that this model allows loops and repeated edges). In what follows, we use G_t to denote the graph induced by the first t edges, and given the orientation of each edge, use D_t to denote the directed graph induced by the first t edges. By m_* we denote the time t when the last vertex of degree one in G_t becomes a degree two vertex.

We will first prove that there exists an on-line algorithm **Orient** which *whp* orients the edges of the graph G_{m_*} so that the directed graph D_{m_*} is Hamiltonian, and then in Section 6 show how Theorem 1.1 can be recovered from this result.

1.3 Organization of the Paper

In the next section we describe the algorithm **Orient** that is used to prove Theorem 1.1 (in the modified model). Then in Section 3 we outline the proof of Theorem 1.1. Section 4 describes several properties that a typical random edge process possesses. Using these properties we prove Theorem 1.1 in Section 5. Then in Section 6, we show how to modify the algorithm **Orient**, in order to make it work for the original random graph process.

Notation. A *directed 1-factor* is a directed graph in which every vertex has in-degree and out-degree exactly 1, and a *1-factor* of a directed graph is a spanning subgraph which is a directed 1-factor. The function $\exp(x) := e^x$ is the exponential function. Throughout the paper $\log(\cdot)$ denotes the natural logarithm. For the sake of clarity, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize our absolute constants. We also assume that the order n of all graphs tends to infinity and therefore is sufficiently large whenever necessary.

2 The Orientation Rule

In this section we describe the algorithm **Orient**. Its input is the edge process $\mathbf{e} = (e_1, e_2, \ldots, e_{m_*})$, and output is an on-line orientation of each edge e_t. The algorithm proceeds in two steps. In the first step, which consists of the first $2n \log\log n$ edges, the algorithm builds a "core" which contains almost all the vertices, and whose edges are distributed (almost) like a 6-in 6-out random graph. In the second step, which contains all edges that follow, the remaining $o(n)$ non-core vertices are taken care of, by being connected to the core in a way that will guarantee *whp* the existence of a directed Hamiltonian cycle.

2.1 Step I

Recall that each edge is given as an ordered pair (v, w). For every vertex v we keep a count of the number of times that v appears as the first vertex. We update the set of *saturated* vertices, which consists of the vertices which appeared at least 12 times as the first vertex. Given the edge (v, w) at time t, if v is still not saturated, direct the edge (v, w) alternatingly with respect to v starting from an out edge (by alternatingly we mean, if the last edge having v as the first vertex was directed as an out edge of v, then direct the current one as an in edge of v, and vice-versa. For the first edge we choose arbitrarily the out direction). Otherwise, if v is saturated, then count the number of times that w appeared as a second vertex when the first vertex is already saturated, and direct the edges alternatingly according to this count with respect to w starting from an in edge. This alternation process is independent to the previous one. That is, even if w appeared as a first vertex somewhere before, the count should be kept track separately from it.

For a vertex $v \in V$, let the *first vertex degree* of v be the number of times that v appeared as a first vertex in Step I, and denote it as $d_1(v)$. Let the *second vertex degree* of v be the number of times that v appeared in Step I as a second vertex of an edge whose first vertex is already saturated, and denote it as $d_2(v)$. Note that the sum of the first vertex degree and second vertex degree of v is not necessarily equal to the degree of v in Step I as v might appear as a second vertex of an edge whose first vertex is not yet saturated. We will call such an edge a *neglected edge* of v.

2.2 Step II

Let A be the set of saturated vertices at the end of Step I, and $B = V \setminus A$. Call an edge an A-B edge if one end point lies in A and the other end point lies in B, and similarly define A-A edges and B-B edges. Given an edge $e = (v, w)$ at time t, if e is an A-B edge, and w.l.o.g. assume that $v \in B$ and $w \in A$, then direct e alternatingly with respect to v, where the alternation process of Step II continues the one from Step I as follows:

1. If v appeared as a first vertex in Step I at least once, then pick up where the alternation process of v as a first vertex in Step I stopped and continue the alternation.
2. If v did not appear as a first vertex in Step I but did appear as a second vertex of an already saturated vertex, then pick up where the alternation process of v as a second vertex of a saturated vertex stopped in Step I and continue the alternation.
3. If v appeared in Step I but does not belong to the above two cases, then consider the first neglected edge connected to v, and start the alternation process from the opposite direction of this edge.
4. If none of the above, then start from an out edge.

Otherwise, if e is an A-A edge or a B-B edge, orient it uniformly at random. Note that unlike Step I, the order of vertices of the given edge does not affect the orientation of the edge in Step II.

For a vertex $v \in B$, let the A-B *degree* of v be the number of A-B edges incident to v in Step II, and denote it as $d_{AB}(v)$. For $v \in A$, let $d_{AB}(v) = 0$.

3 Proof Outline

Our approach builds on Frieze's proof of the Hamiltonicity of the random directed graph process [14] with some additional ideas. His proof consists of two phases (the original proof consists of three phases, but for simplicity, we describe it as two phases). We shall first describe these two phases of Frieze's proof, and then point out the modifications that are necessary to accommodate our different setting. Let $m = (1 + o(1))n \log n$ be the time at which the random directed graph process has minimum in-degree and out-degree 1, and let $D_{n,m}$ be the directed graph at time m (throughout this section we say that random directed graphs have certain properties if they have the properties *whp*).

3.1 Phase 1 : Find a small 1-factor

In Phase 1, a 1-factor of $D_{n,m}$ consisting of at most $O(\log n)$ cycles is constructed. To this end, a subgraph $D_{5-in,5-out}$ of $D_{n,m}$ is constructed which uses only a small number of the edges. Roughly speaking, for each vertex, use its first 5 out-neighbors and 5 in-neighbors (if possible) to construct $D_{5-in,5-out}$. Note that the resulting graph will be similar to a random 5-in 5-out directed graph, but still different as some vertices will only have 1 in-neighbor and 1 out-neighbor even at time m. Finally, viewing $D_{5-in,5-out}$ as a bipartite graph $G'(V \cup V^*, E')$, where V^* is a copy of V, and $\{u, v^*\} \in E'$ iff $u \to v$ belongs to $D_{5-in,5-out}$, one proves that G' has a perfect matching. It turns out that this matching can be viewed as a uniform random permutation of the set of vertices V. A well known fact about such permutations is that they *whp* consist of at most $O(\log n)$ cycles.

3.2 Phase 2 : Combining the cycles into a Hamilton cycle

In Phase 2, the cycles of the 1-factor are combined into a Hamilton cycle. The technical issue to overcome in this step is the fact that in order to construct $D_{5-in,5-out}$, all of the edges were scanned, and now supposedly we have no remaining random edges in the process to combine the cycles of the 1-factor. However, note that since $D_{5-in,5-out}$ consists of at most $10n$ edges, the majority of edges need not be exposed. More rigorously, let $LARGE$ be the vertices whose degree is $\Omega(\log n/\log\log n)$ at time $t_0 = 2n\log n/3$ in the directed graph process. For the $LARGE$ vertices, its 5 neighbors in $D_{5-in,5-out}$ will be determined solely by the edges up to time t_0, leaving the remaining edges (edges after time t_0) of the process unexposed. Two key properties used in Phase 2 are that *whp*, (a) $|LARGE| = n - o(n^{1/2})$, and (b) every cycle of the 1-factor contains many $LARGE$ vertices. Note that by (a), out of the remaining $n\log n/3$ edges, all but $o(1)$-fraction will connect two $LARGE$ vertices. Phase 2 can now be summarized by the following theorem [14].

Theorem 3.1. *Let V be a set of n vertices and $L \subset V$ be a subset of size at least $n - o(n^{1/2})$. Assume that D is a directed 1-factor over V consisting of at most $O(\log n)$ cycles, and the vertices $V \setminus L$ are at distance at least 10 away from each other in this graph.*

If $(1/3 - o(1))n\log n$ L-L edges are given uniformly at random, then whp the union of these edges and the graph D contains a directed Hamilton cycle.

The proof of a slightly stronger version of Theorem 3.1 will be given in Section 6.

3.3 Comparing with our setting

The main technical issue in this paper is to reprove Phase 1, namely, the existence of a 1-factor with small number of cycles. In [14], the fact that all vertices have the same distribution in $D_{5-in,5-out}$, led to an argument showing the existence of a matching that translates into a uniform random permutation. Our case is different because of the orientation rule. We have different types of vertices each being oriented in a different way, breaking the nice symmetry. The bulk of our technical work is spent in resolving this technical issue.

Once this is done, that is after achieving the 1-factor, we come up with an analogue of $LARGE$, which we call "saturated". Similarly as in Phase 2 described above, we prove that *whp* (a') most of the vertices are saturated, and (b') every cycle in the 1-factor contains many saturated vertices. However, the naive approach results in a situation where one cannot apply Theorem 3.1 ((a') and (b') are quantitatively weaker than (a) and (b)). Thus we develop the argument of "compressing" vertices of a given cycle. This idea allows us to get rid of all the non-saturated vertices, leading to another graph which only has saturated vertices in it. Details will be given in Section 5.2. Once we apply the compression argument, we can use Theorem 3.1 to finish the proof. Let us mention that the compression argument can be applied after Phase 1 in [14] as well to simplify the proof.

4 A Typical Random Process

The following well-known concentration result (see, for example [1, Corollary A.1.14]) will be used several times in the proof. We denote by $Bi(n,p)$ the binomial random variable with parameters n and p.

Theorem 4.1. *(Chernoff's inequality) If $X \sim Bi(n,p)$ and $\varepsilon > 0$, then*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]\big) \leq e^{-\Omega_\varepsilon(\mathbb{E}[X])}.$$

4.1 Classifying Vertices

To analyze the algorithm it will be convenient to work with three sets of vertices. The first is the set of **saturated** vertices at Step I. Throughout we will use A to denote this set. Let us now consider the non-saturated vertices $B = V \setminus A$. Here we distinguish between two types. We say that $v \in B$ **blossoms** if there are at least 12 edges of the form $\{v, A\}$ in Step II (by A we mean an arbitrary vertex from A), and let B_1 be the collection of vertices which blossom. All the remaining vertices are **restricted**, and is denoted by B_2. Thus every vertex either is saturated (A), blossoms (B_1), or is restricted (B_2).

Furthermore, the set of restricted vertices has two important subclasses which are determined by the first vertex degree $d_1(v)$, second vertex degree $d_2(v)$, and A-B degree $d_{AB}(v)$ defined in the previous section. We say that a restricted vertex v **partially-blossoms** if the sum of its first vertex degree, second vertex degree, and A-B degree is at least 2. Note that since we stopped the process when the graph has minimum degree 2, every vertex v has degree at least 2. Thus, if the above mentioned sum is at most 1, then v either has a neglected edge, or a B-B edge connected to it. A useful fact that we prove in Lemma 4.5 says that *whp* all such vertices v have one A-B edge (thus $d_{AB}(v) = 1$), and at least one neglected edge. Thus, we call a restricted vertex v not being partially-blossomed, and having one A-B edge and at least one neglected edge as a **bud**.

4.2 Properties of a Typical Random Process

In this section we list several properties that hold *whp* for random edge processes. We will call an edge process **typical** if indeed the properties hold. Let

$$m_1 = \frac{1}{2}n\log n + \frac{1}{2}n\log\log n - n\log\log\log n, \qquad m_2 = \frac{1}{2}n\log n + \frac{1}{2}n\log\log n + n\log\log\log n.$$

Note that for a fixed vertex v, the probability of an edge being incident to v is $\frac{2n-1}{n^2} = \frac{2}{n} - \frac{1}{n^2}$ (this is because in our process, each edge is given by an ordered pair of vertices). However as it turns out the small order term $\frac{1}{n^2}$ is always negligible for our purpose, so we will use the probability $\frac{2}{n}$ for this event, and remind the reader that the term $\frac{1}{n^2}$ is omitted. Recall that the stopping time m_* is the time at which the last vertex of degree one becomes a degree two vertex and the process stops.

Claim 4.2. *Let m_* be the stopping time of the random process. Then whp*

$$m_1 \leq m_* \leq m_2.$$

Proof. For a fixed vertex v, the probability of an edge being incident to v is about $\frac{2}{n}$. Hence the probability of v having degree at most 1 at time m_2 is,

$$\left(1 - \frac{2}{n}\right)^{m_2} + \binom{m_2}{1}\frac{2}{n} \cdot \left(1 - \frac{2}{n}\right)^{m_2 - 1} \leq 3\log n \cdot e^{-\log n - \log\log n - 2\log\log\log n} = O\left(\frac{1}{n(\log\log n)^2}\right).$$

Thus by Markov's inequality, *whp* there is no vertex of degree at most 1 after m_2 edges. This shows that $m_* \leq m_2$. Similarly, the expected number of vertices having degree at most 1 after seeing m_1

edges is $\Omega((\log\log n)^2)$, and by computing the second moment of the number of vertices having degree at most 1, we can show that after m_1 edges *whp* at least one such vertex exits. This shows that $m_* \geq m_1$. The rest of the details are fairly standard and are omitted. $\square$

Next we are going to list some properties regarding the different types of vertices.

Claim 4.3. *The number of saturated vertices satisfies whp*

$$|A| \geq n\left(1 - \frac{(\log\log n)^{12}}{\log^2 n}\right).$$

Proof. For a fixed vertex v, the probability of v occurring as the first vertex of an edge is (exactly) $\frac{1}{n}$, and thus the probability of v ending up non-saturated at Step I is at most

$$\sum_{k=0}^{11}\binom{2n\log\log n}{k}\left(\frac{1}{n}\right)^k \cdot \left(1-\frac{1}{n}\right)^{2n\log\log n - k} \leq \sum_{k=0}^{11}(2\log\log n)^k \frac{1}{\log^2 n} = O\left(\frac{(\log\log n)^{11}}{\log^2 n}\right).$$

The claim follows from Markov's inequality. $\square$

Our next goal is to prove that the restricted vertices consist only of partially-blossomed and bud vertices. For that we need the following auxiliary lemma.

Claim 4.4. *Let E_{BB} be the collection of all B-B edges (in Step II). The graph $G_{m_*} \setminus E_{BB}$ has whp minimum degree 2.*

Proof. If the graph $G_{m_*} \setminus E_{BB}$ has minimum degree less than 2 for some edge process $\mathbf{e}$, then there exists a vertex v which gets at most one edge other than a B-B edge, and at least one B-B edge. By Claim 4.2, it suffices to prove that the graph *whp* does not contain a vertex which has at most one edge other than a B-B edge at time m_1, and at least one B-B edge at time m_2. Let $\mathcal{A}_v$ be the event that v is such vertex. Let $\mathcal{BS}$ be the event that $|B| \leq \frac{(\log\log n)^{12}}{\log^2 n} n$ (B is small), and note that $\mathbb{P}(\mathcal{BS}) = 1 - o(1)$ by Claim 4.3. Then we have

$$\mathbb{P}(G_{m_*} \setminus E_{BB} \text{ has minimum degree less than 2}) = \mathbb{P}\left(\bigcup_{v\in V}\mathcal{A}_v\right) \leq n \cdot \mathbb{P}\left(\mathcal{A}_v \cap \mathcal{BS}\right) + o(1). \tag{1}$$

The event $\mathcal{A}_v$ is equivalent to the vertex v receiving k B-B edges, for some $k > 0$, and at most one edge other than a B-B edge at appropriate times. This event is contained in the event $\mathcal{C}_v \cap \mathcal{D}_{v,k}$ where $\mathcal{C}_v$ is the event "v appears at most once in Step I", and $\mathcal{D}_{v,k}$ is the event "$d_{AB}(v) \leq 1$ by time m_1 and v receives k B-B edges by time m_2". Therefore our next goal is to bound

$$\mathbb{P}(\mathcal{C}_v \cap \mathcal{D}_{v,k} \cap \mathcal{BS}) = \mathbb{P}(\mathcal{C}_v \cap \mathcal{BS}) \cdot \mathbb{P}(\mathcal{D}_{v,k}|\mathcal{C}_v \cap \mathcal{BS}) \leq \mathbb{P}(\mathcal{C}_v) \cdot \mathbb{P}(\mathcal{D}_{v,k}|\mathcal{C}_v \cap \mathcal{BS}). \tag{2}$$

We can bound the probability of the event $\mathcal{C}_v$ by,

$$\left(1-\frac{2}{n}\right)^{2n\log\log n} + \binom{2n\log\log n}{1}\left(\frac{2}{n}\right) \cdot \left(1-\frac{2}{n}\right)^{2n\log\log n - 1} = O\left(\frac{\log\log n}{\log^4 n}\right). \tag{3}$$

To bound the event $\mathcal{D}_{v,k}$ which is "$d_{AB}(v) \leq 1$ at time m_1 and v receives k B-B edges by time m_2", note that $\mathcal{C}_v$ and $\mathcal{BS}$ are events which depend only on the first $2\log\log n$ edges (Step I edges).

Therefore conditioning on this event does not affect the distribution of edges in Step II (each edge is chosen uniformly at random among all possible n^2 pairs). We only consider the case $d_{AB}(v) = 1$ (the case $d_{AB}(v) = 0$ can be handled similarly, and turns out to be dominated by the case $d_{AB}(v) = 1$). Thus to bound the probability, we choose $k+1$ edges among the $m_2 - 2n \log\log n$ edges, let 1 of them to be an A-B edge, k of them to be B-B edges incident to v. Moreover, since $d_{AB}(v) \le 1$ at time m_1, we know that at least $m_1 - 2n \log\log n - k - 1$ edges are not incident to v. Thus,

$$\begin{aligned}&\mathbb{P}(\mathcal{D}_{v,k} \,|\, \mathcal{C}_v \cap \mathcal{BS}) \\ &\quad\le \binom{m_2 - 2n\log\log n}{k+1} \left(\frac{2}{n}\right)^{k+1} \binom{k+1}{1} \frac{|A|}{n} \left(\frac{|B|}{n}\right)^k \left(1 - \frac{2}{n}\right)^{m_1 - 2n\log\log n - k - 1}.\end{aligned}$$

By using the inequalities $1 - x \le e^{-x}$, $|A| \le n$, and $\binom{m_2 - 2n\log\log n}{k+1} \le m_2^{k+1}$, the probability above is bounded by

$$(k+1) m_2^{k+1} \left(\frac{2}{n}\right)^{k+1} \left(\frac{|B|}{n}\right)^k \exp\left(-\frac{2}{n}(m_1 - 2n\log\log n - k - 1)\right). \tag{4}$$

Therefore by (2), (3), and (4),

$$\begin{aligned}&\mathbb{P}(\mathcal{C}_v \cap \mathcal{D}_{v,k} \cap \mathcal{BS}) \le \\ &O\left(\frac{\log\log n}{\log^4 n}\right)(k+1) m_2^{k+1} \left(\frac{2}{n}\right)^{k+1} \left(\frac{|B|}{n}\right)^k \exp\left(-\frac{2}{n}(m_1 - 2n\log\log n - k - 1)\right).\end{aligned}$$

Plugging the bound $|B| \le \frac{n \log\log^{12} n}{\log^2 n}$ and $m_2 \le n \log n$ in the latter, one obtains:

$$O(k) \left(\frac{\log\log n}{\log^3 n}\right) \left(\frac{2(\log\log n)^{12}}{\log n}\right)^k \exp\left(-\frac{2}{n}(m_1 - 2n\log\log n - k - 1)\right).$$

By the definition $m_1 = \frac{1}{2} n \log n + \frac{1}{2} n \log\log n - n \log\log\log n$, this further simplifies to

$$O(k) \left(\frac{(\log\log n)^3}{n}\right) \left(\frac{2e^{2/n}(\log\log n)^{12}}{\log n}\right)^k.$$

Summing over all possible values of k,

$$\sum_{k=1}^{\infty} \mathbb{P}(\mathcal{C}_v \cap \mathcal{D}_{v,k} \cap \mathcal{BS}) \le \sum_{k=1}^{\infty} \frac{O(k)(\log\log n)^3}{n} \left(\frac{4(\log\log n)^{12}}{\log n}\right)^k = o(n^{-1}).$$

Going back to (1), we get that

$$\mathbb{P}(G_{m_*} \setminus E_{BB} \text{ has minimum degree less than 2}) = n \cdot o(n^{-1}) + o(1) = o(1).$$

Note that as mentioned in the beginning of this section, we used $\frac{2}{n}$ to estimate the probability of an edge being incident to a fixed vertex. This probability is in fact $\frac{2}{n} - \frac{1}{n^2}$, but the term $\frac{1}{n^2}$ will only affect the lower order estimates. $\square$

Claim 4.5. *Every restricted vertex is whp either partially-blossomed, or a bud.*

Proof. Assume there exists a restricted vertex v which is not partially-blossomed or a bud. Then by definition, the sum $d_1(v)+d_2(v)+d_{AB}(v) \leq 1$. The possible values of the degrees $(d_1(v), d_2(v), d_{AB}(v))$ are $(1,0,0), (0,1,0), (0,0,1)$, or $(0,0,0)$. Vertices which correspond to $(0,0,1)$ will all be bud vertices *whp* by Claim 4.4. It suffices to show then that *whp* there does not exist vertices which correspond to $(1,0,0), (0,1,0)$, or $(0,0,0)$. Let T be the collection of vertices which have $d_1(v)+d_2(v) \leq 1$ and $d_{AB}(v) = 0$ at time m_1. By Claim 4.2 it suffices to prove that T is empty. Let $\mathcal{BS}$ be the event $|B| \leq \frac{(\log\log n)^{12}}{\log^2 n} n$, and note that by Claim 4.3, $\mathbb{P}(\mathcal{BS}) = 1 - o(1)$. The event $\{T \neq \emptyset\}$ is the same as $\cup_{v \in V}\{v \in T\}$, and thus by the union bound,

$$\begin{aligned}\mathbb{P}(T \neq \emptyset) &\leq o(1) + \sum_{v \in V} \mathbb{P}(\{v \in T\} \cap \mathcal{BS}) \\ &= o(1) + \sum_{v \in V} \mathbb{P}\left(\{d_1(v)+d_2(v) \leq 1\} \cap \{d_{AB}(v) = 0\} \cap \mathcal{BS}\right).\end{aligned}$$

By Bayes equation, the second term of right hand side splits into,

$$\begin{aligned}&\sum_{v \in V} \mathbb{P}\left(\{d_1(v)+d_2(v) \leq 1\} \cap \mathcal{BS}\right) \cdot \mathbb{P}\left(d_{AB}(v) = 0 \,|\, \{d_1(v)+d_2(v) \leq 1\} \cap \mathcal{BS}\right) \\ &\quad \leq \sum_{v \in V} \mathbb{P}\left(d_1(v)+d_2(v) \leq 1\right) \cdot \mathbb{P}\left(d_{AB}(v) = 0 \,|\, \{d_1(v)+d_2(v) \leq 1\} \cap \mathcal{BS}\right). \qquad (5)\end{aligned}$$

The probability $\mathbb{P}(d_1(v)+d_2(v) \leq 1)$ can be bounded by $\mathbb{P}(\{d_1(v) \leq 1\} \cap \{d_2(v) \leq 1\})$ which satisfies,

$$\mathbb{P}(\{d_1(v) \leq 1\} \cap \{d_2(v) \leq 1\}) = \mathbb{P}(d_1(v) \leq 1) \cdot \mathbb{P}(d_2(v) \leq 1 \,|\, d_1(v) \leq 1).$$

The term $\mathbb{P}(d_1(v) \leq 1)$ can be easily calculated as,

$$\left(1 - \frac{1}{n}\right)^{2n\log\log n} + \binom{2n\log\log n}{1}\left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n}\right)^{2n\log\log n - 1} = O\left(\frac{\log\log n}{\log^2 n}\right).$$

To estimate $\mathbb{P}(d_2(v) \leq 1 \,|\, d_1(v) \leq 1)$, expose the edges of Step I as follows: First expose all the first vertices. Then expose the second vertices whose first vertex is saturated ($d_2(v)$ is now determined for every $v \in V$). The number of second-vertex-spots that are considered is at least $2n\log\log n - 12n$, and thus $\mathbb{P}(d_2(v) \leq 1 | d_1(v) \leq 1)$ is at most

$$\left(1 - \frac{1}{n}\right)^{2n\log\log n - 12n} + \binom{2n\log\log n}{1}\left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n}\right)^{2n\log\log n - 12n - 1} = O\left(\frac{\log\log n}{\log^2 n}\right).$$

Thus as a crude bound, we have

$$\mathbb{P}(d_1(v)+d_2(v) \leq 1) \leq \mathbb{P}(d_1(v) \leq 1) \cdot \mathbb{P}(d_2(v) \leq 1 \,|\, d_1(v) \leq 1) = O\left(\frac{(\log\log n)^2}{\log^4 n}\right).$$

Since $d_1(v)+d_2(v) \leq 1$ implies that $v \in B$, and $d_{AB}(v)$ depends only on the Step II edges (which are independent from $d_1(v)$, $d_2(v)$, and $\mathcal{BS}$), the second term of the right hand side of equation (5), the probability $\mathbb{P}\left(d_{AB}(v) = 0 \,|\, \{d_1(v) + d_2(v) \le 1\} \cap \mathcal{BS}\right)$ can be bounded by

$$\begin{aligned}
\left(1 - 2\frac{1}{n}\frac{|A|}{n}\right)^{m_1 - 2n\log\log n} &\le \exp\left(-2(m_1 - 2n\log\log n)|A|/n^2\right) \\
\le \exp\left(-(\log n - 3\log\log n - 2\log\log\log n)\left(1 - \frac{(\log\log n)^{12}}{\log^2 n}\right)\right) \\
\le \exp\left(-\log n + 3\log\log n + 2\log\log\log n + o(1)\right) = O\left(\frac{(\log n)^3(\log\log n)^2}{n}\right).
\end{aligned}$$

Therefore in (5),

$$\begin{aligned}
\mathbb{P}(T \ne \emptyset) &\le o(1) + \sum_{v \in V} O\left(\frac{(\log\log n)^2}{\log^4 n}\right) O\left(\frac{(\log n)^3(\log\log n)^2}{n}\right) \\
&= o(1) + O\left(\frac{(\log\log n)^4}{\log n}\right) = o(1).
\end{aligned}$$

□

Claim 4.6. *The following properties hold whp for* restricted *vertices:*

(i) There are at most $\log^{13} n$ *such vertices,*

(ii) every such two vertices are at distance at least 3 in G_{m_*} *from each other.*

Proof. Since being a restricted vertex is a monotone decreasing property, by Claim 4.2 it suffices to prove (i) at time m_1. Recall that B_2 is the collection of restricted vertices (a vertex is restricted if it is not saturated or blossomed).

First, condition on the whole outcome of Step I edges (first $2n\log\log n$ edges) and the event that $|B| \le \frac{(\log\log n)^{12}}{\log^2 n} n$. Then the set B is determined, and for a vertex $v \in B$, we can bound the probability of the event $v \in B_2$ as following

$$\mathbb{P}\left(v \in B_2\right) \le \sum_{\ell=0}^{11} \binom{m_2}{\ell} \left(\frac{2}{n}\right)^{\ell} \left(1 - \frac{2|A|}{n^2}\right)^{m_1 - 2n\log\log n - \ell}. \tag{6}$$

Use the inequalities $m_1 = \frac{1}{2}n\log n + \frac{1}{2}n\log\log n - \log\log\log n \le n\log n$, $m_2 \le n\log n$, $1 - x \le e^{-x}$, and $|A| = n - |B| \ge n\left(1 - \frac{(\log\log n)^{12}}{\log^2 n}\right)$ to bound the above by

$$\sum_{l=0}^{11} (2\log n)^{\ell} \exp\left(-(\log n - 3\log\log n - 2\log\log\log n - \ell)\left(1 - \frac{(\log\log n)^{12}}{\log^2 n}\right)\right).$$

The sum is dominated by $\ell = 11$, and this gives

$$O\left(\log^{11} n\right) \exp\left(-\log n + 3\log\log n + 2\log\log\log n + o(1)\right) \le O\left(\frac{(\log\log n)^2 \log^{14} n}{n}\right).$$

Thus the expected size of B_2 given the Step I edges is

$$\mathbb{E}[|B_2| \,|\, \text{Step I edges}] \le |B| \cdot O\left(\frac{(\log\log n)^2 \log^{14} n}{n}\right) \le O((\log\log n)^{14} \log^{12} n).$$

Since the assumptions on A and B holds *whp* by Claim 4.3, we can use Markov inequality to conclude that *whp* there are at most $\log^{13} n$ vertices in B_2. Let us now prove (ii).

For three distinct vertices v_1, v_2 and w in V, let $\mathcal{A}(v_1, v_2, w)$ be the event that w is a common neighbor of v_1 and v_2. The probability of there being edges (v_1, w) (or (w, v_1)) and (v_2, w) (or (w, v_2)) and $v_1, v_2 \in B_2$ can be bounded by first choosing two time slots where (v_1, w) (or (w, v_1)) and (v_2, w) (or (w, v_2)) will be placed, and then filling in the remaining edges so that $v_1, v_2 \in B_2$. We will only bound the event of there being edges (v_1, w) and (w, v_2) in the edge process (other cases can be handled in a similar manner). The probability we would like to bound is

$$\mathbb{P}(\exists v_1, v_2, w, \exists 1 \le t_1, t_2 \le m_2, e_{t_1} = (v_1, w), e_{t_2} = (w, v_2), v_1, v_2 \in B_2).$$

By the union bound this probability is at most

$$\begin{aligned}
&\sum_{v_1, v_2, w \in V} \sum_{t_1, t_2 = 1}^{m_2} \mathbb{P}(e_{t_1} = (v_1, w), e_{t_2} = (w, v_2), v_1, v_2 \in B_2) && (7)\\
=&\sum_{v_1, v_2, w \in V} \sum_{t_1, t_2 = 1}^{m_2} \mathbb{P}(e_{t_1} = (v_1, w), e_{t_2} = (w, v_2))\mathbb{P}(v_1, v_2 \in B_2 | e_{t_1} = (v_1, w), e_{t_2} = (w, v_2))\\
\le&\frac{1}{n} \sum_{t_1, t_2 = 1}^{m_2} \mathbb{P}(v_1, v_2 \in B_2 | e_{t_1} = (v_1, w), e_{t_2} = (w, v_2)). && (8)
\end{aligned}$$

To simplify the notation we abbreviate $\mathbb{P}(v_1, v_2 \in B_2 | e_{t_1} = (v_1, w), e_{t_2} = (w, v_2))$ by $\mathbb{P}(v_1, v_2 \in B_2 | e_{t_1}, e_{t_2})$. By using the independence of Step I and Step II edges we have,

$$\mathbb{P}(v_1, v_2 \in B_2 | e_{t_1}, e_{t_2}) = \mathbb{P}(v_1, v_2 \in B | e_{t_1}, e_{t_2})\mathbb{P}(v_1, v_2 \notin B_1 | v_1, v_2 \in B, e_{t_1}, e_{t_2}).$$

For fixed t_1 and t_2, we can bound $\mathbb{P}(v_1, v_2 \in B | e_{t_1}, e_{t_2})$ by the probability of "v_1 and v_2 appear at most 22 times combined in Step I as a first vertex other than at time t_1 and t_2", whose probability can be bounded as follows regardless of the value of t_1 and t_2,

$$\sum_{k=0}^{22} \binom{2n \log\log n}{k} \left(\frac{2}{n}\right)^k \cdot \left(1 - \frac{2}{n}\right)^{2n \log\log n - 2 - k} \le \sum_{k=0}^{22} (4 \log\log n)^k \frac{O(1)}{\log^4 n} = O\left(\frac{(\log\log n)^{22}}{\log^4 n}\right).$$

To bound $\mathbb{P}(v_1, v_2 \notin B_1 | v_1, v_2 \in B, e_{t_1}, e_{t_2})$, it suffices to bound $\mathbb{P}(v_1, v_2 \notin B_1 | v_1, v_2 \in B, e_{t_1}, e_{t_2}, \mathcal{BS})$, which can be bounded by the probability of "v_1 and v_2 receives at most 22 A-B edges combined in Step II other than at time t_1 and t_2". Regardless of the value of t_1 and t_2, this satisfies the bound,

$$\sum_{\ell=0}^{22} \binom{m_2}{\ell} \left(\frac{4}{n}\right)^\ell \left(1 - \frac{4}{n}\frac{|A|}{n}\right)^{m_1 - 2 - 2n \log\log n - l}.$$

Note that $\frac{4}{n}$ and $\frac{2}{n}$ in this equation should in fact involve some terms of order $\frac{1}{n^2}$, but we omitted it for simplicity since it does not affect the asymptotic final outcome. By a similar calculation to (6), this eventually can be bounded by $O(\frac{\log^{29} n}{n^2})$. Thus we have

$$\mathbb{P}(v_1, v_2 \in B_2 | e_{t_1}, e_{t_2}) = O\left(\frac{\log^{26} n}{n^2}\right),$$

which by (8) and $m_2 \le n \log n$ gives,

$$\mathbb{P}(\exists v_1, v_2, w, \exists 1 \le t_1, t_2 \le m_2, e_{t_1} = (v_1, w), e_{t_2} = (w, v_2), v_1, v_2 \in B_2) \le O\left(\frac{\log^{28} n}{n}\right).$$

Therefore by Markov's inequality, *whp* no such three vertices exist, which implies that two vertices $v_1, v_2 \in B_2$ cannot be at distance two from each other in G_{m_*}. Similarly, we can prove that *whp* every two vertices $v_1, v_2 \in B_2$ are not adjacent to each other, and hence *whp* every $v_1, v_2 \in B_2$ are at distance at least two away from each other. □

4.3 Configuration of the edge process

To prove that our algorithm succeeds *whp*, we first reveal some pieces of information of the edge process, which we call the "configuration" of the process. These information will allow us to determine whether the underlying edge process is typical or not. Then in the next section, using the remaining randomness, we will construct a Hamilton cycle.

In the beginning, rather than thinking of edges coming one by one, we regard our edge process $\mathbf{e} = (e_1, e_2, \cdots, e_{m_*})$ as a collection of edges e_i for $i = 1, \cdots, m_*$ whose both endpoints are not known. We can decide to reveal certain information as necessary. Let us first reveal the following.

1. For $t \le 2n \log\log n$, reveal the first vertex of the t-th edge e_t. If this vertex already appeared as the first vertex at least 12 times among the edges $e_1, \cdots, e_{t-1}$, then also reveal the second vertex.

Given this information, we can determine the saturated vertices, and hence we know the sets A and B. Therefore, it is possible to reveal the following information.

2. For $t > 2n \log\log n$, reveal all the vertices that belong to B.

The information we revealed determines the blossomed (B_1), and restricted (B_2) vertices. Thus we can further reveal the following information.

3. For $t \le 2n \log\log n$, further reveal all the non-revealed vertices that belong to B_2.
4. For every edge $e_t = (v_t, w_t)$ in which we already know that either $v_t \in B_2$ or $w_t \in B_2$, also reveal the other vertex.

We define the *configuration* of an edge process as the above four pieces of information.

We want to say that all the non-revealed vertices are uniformly distributed over certain sets. But in order for this to be true, we must make sure that the distribution of the non-revealed vertices is not affected by the fact that we know the value of m_* (some vertex has degree exactly 2 at time m_*, and maybe a non-revealed vertex will make this vertex to have degree 2 earlier than m_*). This is indeed the case, since the last vertex to have degree 2 is necessarily a restricted vertex, and all the locations of the restricted vertices are revealed. Thus the non-revealed vertices cannot change the value of m_*. Therefore, once we condition on the configuration of an edge process, the remaining vertices are distributed in the following way:

(i) For $t \leq 2n \log\log n$, if the first vertex of the edge e_t appeared at most 12 times among $e_1, \cdots, e_{t-1}$, then its second vertex is either a known vertex in B_2 or is a random vertex in $V \setminus B_2$.

(ii) For $t > 2n \log\log n$, if both vertices of e_t are not revealed, then e_t consists of two random vertices of A. If only one of the vertices of e_t is not revealed, then the revealed vertex is in B, and the non-revealed vertex is a random vertex of A.

Definition 4.7. *A configuration of an edge process is* typical *if it satisfies the following.*

(i) The number of saturated and blossomed vertices satisfy $|A| \geq n - \frac{(\log\log n)^{12}}{\log^2 n} n$, *and* $|B_1| \leq \frac{(\log\log n)^{12}}{\log^2 n} n$ *respectively.*

(ii) The number of restricted vertices satisfies $|B_2| \leq \log^{13} n$.

(iii) Every vertex appears at least twice in the configuration even without considering the B-B *edges.*

(iv) All the restricted vertices are either partially-blossomed or buds.

(v) In the non-directed graph induced by the edges whose both endpoints are revealed, every two restricted vertices v_1, v_2 *are at distance at least 3 away from each other.*

(vi) There are at least $\frac{1}{3} n \log n$ *edges* e_t *for* $t > 2n \log\log n$ *whose both endpoints are not yet revealed.*

Lemma 4.8. *The random edge process has a typical configuration whp.*

Proof. The fact that the random edge process has *whp* a configuration satisfying $(i), (iii)$, and (iv) follows from Claims 4.3, 4.4, and 4.5 respectively. (ii) and (v) follow from Claim 4.6. To verify (vi), note that by Claim 4.2 and 4.3, *whp* there are at least $\frac{1}{2} n \log n - 2n \log\log n$ edges of Step II, and $|A| = (1 - o(1))n$. Therefore the probability of a Step II edge being an A-A edge is $1 - o(1)$, and the expected number of A-A edges is $(1/2 - o(1)) n \log n$. Then by Chernoff's inequality, *whp* there are at least $\frac{1}{3} n \log n$ A-A edges. These edges are the edges we are looking for in (vi). $\square$

5 Finding a Hamilton Cycle

In the previous section, we established several useful properties of the underlying graph G_{m_*}. In this section, we study the algorithm **Orient** using these properties, and prove that conditioned on the edge process having a typical configuration, the graph D_{m_*} *whp* contains a Hamilton cycle (recall that the graph D_{m_*} is the set of random edges of the edge process, oriented according to **Orient**). As described in Section 3, the proof is a constructive proof, in the sense that we describe how to find such a cycle. The algorithm is similar to that used in [14] which we described in some details in Section 3. Let us briefly recall that it proceeds in two stages:

1. Find a 1-factor of G. If it contains more than $O(\log n)$ cycles, fail.
2. Join the cycles into a Hamilton cycle.

The main challenge in our case is to prove that the first step of the algorithm does not fail. Afterwards, we argue why we can apply Frieze's results for the remaining step.

5.1 Almost 5-in 5-out subgraph

Let $D_{5-in,5-out}$ be the following subgraph of D_{m_*}. For each vertex v, assign a set of neighbors $OUT(v)$ and $IN(v)$, where $OUT(v)$ are out-neighbors of v and $IN(v)$ are in-neighbors of v. For saturated and blossomed vertices, $OUT(v)$ and $IN(v)$ will be of size 5, and for restricted vertices, they will be of size 1 (thus $D_{5-in,5-out}$ is not a 5-in 5-out directed graph under the strict definition).

Let E_1 be the edges of Step I (first $2n \log\log n$ edges), and E_2 be the edges of Step II (remaining edges).

- If v is saturated, then consider the first 12 appearances in E_1 of v as a first vertex. Some of these edges might later be used as OUT or IN for other vertices. Hence among these 12 appearances, consider only those whose second vertex is not in B_2. By property (v) of Definition 4.7, there will be at least 11 such second vertices for a typical configuration. Define $OUT(v)$ as the first 5 vertices among them which were directed out from v, and $IN(v)$ as the first 5 vertices among them which were directed in to v in **Orient**.
- If v blossoms, then consider the first 10 A-B edges in E_2 connected to v, and look at the other end points. Let $OUT(v)$ be the first 5 vertices which are an out-neighbor of v and $IN(v)$ be the first 5 vertices which are an in-neighbor of v.

A partially-blossomed vertex, by definition, has $d_1(v)+d_2(v)+d_{AB}(v) \geq 2$, and must fall into one of the following categories. (i) $d_1(v) \geq 2$, (ii) $d_2(v) \geq 2$, (iii) $d_{AB}(v) \geq 2$, (iv) $d_1(v) = 1$, $d_2(v) = 1$, (v) $d_1(v) = 1$, $d_{AB}(v) = 1$, and (vi) $d_1(v) = 0, d_2(v) = 1$, $d_{AB}(v) = 1$. If it falls into several categories, then pick the first one among them.

- If v partially-blossoms and $d_1(v) \geq 2$, consider the first two appearances of v in E_1 as a first vertex. The first is an out-edge and the second is an in-edge (see Section 2.1).
- If v partially-blossoms and $d_2(v) \geq 2$, consider the first two appearances of v in E_1 as a second vertex whose first vertex is saturated. The first is an in-edge and the second is an out-edge (see Section 2.1).
- If v partially-blossoms and $d_{AB}(v) \geq 2$, consider the first two A-B edges in E_2 incident to v. One of it is an out-edge and the other is an in-edge. Note that unlike other cases, the actual order of in-edge and out-edge will depend on the configuration. But since the configuration contains all the positions at which v appeared in the process, the choice of in-edge or out-edge only depends on the configuration and not on the non-revealed vertices (note that this is slightly different from the blossomed vertices).
- If v partially-blossoms and $d_1(v) = 1$, $d_2(v) = 1$, consider the first appearance of v in E_1 as a first vertex, and the first appearance of v in E_1 as a second vertex whose first vertex is saturated. The former is an out-edge and the latter is an in-edge.
- If v partially-blossoms and $d_1(v) = 1$, $d_{AB}(v) = 1$, consider the first appearance of v in E_1 as a first vertex, and the first A-B edge connected to v in E_2. The former is an out-edge and the latter is an in-edge (see rule 1 in Section 2.2).
- If v partially-blossoms and $d_1(v) = 0, d_2(v) = 1$, $d_{AB}(v) = 1$, consider the first appearance of v in E_1 as a second vertex whose first vertex is saturated, and the first A-B edge connected to v in E_2. The former is an in-edge and the latter is an out-edge (see rule 2 in Section 2.2). Thus we can construct $OUT(v)$ and $IN(v)$ of size 1 each, for all partially-blossomed vertices.
- If v is a bud, then consider the first (and only) A-B edge connected to v. Let this edge be e_s. For a typical configuration, by property (iii) of Definition 4.7, we know that v has a neglected edge connected to it. Let e_t be the first neglected edge of v. By property (v) of Definition 4.7, we

know that the first vertex of the neglected edge is either in A or B_1. According to the direction of this edge, the direction of e_s will be chosen as the opposite direction (see rule 3 in Section 2.2). As in the partially-blossomed case with $d_{AB}(v) \geq 2$, the direction is solely determined by the configuration. Thus we can construct $OUT(v)$ and $IN(v)$ of size 1 each (which is already fixed once we fix the configuration).

This in particular shows that D_{m_*} has minimum in-degree and out-degree at least 1, which is clearly a necessary condition for the graph to be Hamiltonian. A crucial observation is that, once we condition on the random edge process having a fixed typical configuration, we can determine exactly which edges are going to be used to construct the graph $D_{5-in,5-out}$ just by looking at the configuration.

For a set X, let $RV(X)$ be an element chosen independently and uniformly at random in the set (consider each appearance of $RV(X)$ as a new independent copy).

Proposition 5.1. *Let $V' = V \setminus B_2$. Conditioned on the edge process having a typical configuration, $D_{5-in,5-out}$ has the following distribution.*

(i) If v is saturated, then $OUT(v)$ and $IN(v)$ are a union of 5 copies of $RV(V')$.
(ii) If v blossoms, then $OUT(v)$ and $IN(v)$ are a union of 5 copies of $RV(A)$.

Proof. For a vertex $v \in V$, the configuration contains the information of the time of arrival of the edges that will be used to construct the set $OUT(v)$ and $IN(v)$.

If v is a saturated vertex, then we even know which edges belong to $OUT(v)$ and $IN(v)$ (if there are no B_2 vertices connected to the first 12 appearances of v as a first vertex, then the first five odd appearances of v as a first vertex will be used to construct $OUT(v)$, and the first five even appearances of v as a first vertex will be used to construct $IN(v)$). Since the non-revealed vertices are independent random vertices in V', we know that $OUT(v)$ and $IN(v)$ of these vertices consist of 5 independent copies of $RV(V')$.

If v blossoms, then the analysis is similar to that of the saturated vertices. However, even though the configuration contains the information of which 10 edges will be used to construct $OUT(v)$ and $IN(v)$, the decision of whether the odd edges or the even edges will be used to construct $OUT(v)$ depends on the particular edge process (this is determined by the orientation rule at Step I). However, since the other endpoints are independent identically distributed random vertices in A, the distribution of $OUT(v)$ and $IN(v)$ is not be affected by the previous edges, and is always $RV(A)$ (this is analogous to the fact that the distribution of the outcome of a coin flip does not depend on whether the initial position was head or tail). $\square$

5.2 A small 1-factor

The main result that we are going to prove in this section is summarized in the following proposition:

Proposition 5.2. *Conditioned on the random edge process having a typical configuration, there exists whp a 1-factor of $D_{5-in,5-out}$ containing at most $2\log n$ cycles, and in which at least $9/10$ proportion of each cycle are saturated vertices.*

Throughout this section, rather than vaguely conditioning on the process having a typical configuration, we will consider a fixed typical configuration $\mathbf{c}$ and condition on the event that the edge process has configuration $\mathbf{c}$. Proposition 5.2 easily follows once we prove that there exists a Hamilton cycle *whp* under this assumption. The reason we do this more precise conditioning is to fix the sets A, B, B_1, B_2 and the edges incident to vertices of B_2 (note that these are determined solely by the configuration). In our later analysis, it is crucial to have these fixed.

To prove Proposition 5.2, we represent the graph $D_{5-in,5-out}$ as a certain bipartite graph in which a perfect matching corresponds to the desired 1-factor of the original graph D_{m^*}. Then using the edge distribution of $D_{5-in,5-out}$ given in the previous section, we will show that the bipartite graph *whp* contains a perfect matching. The proof of Proposition 5.2 will be given at the end after a series of lemmas.

Define a new vertex set $V^* = \{v^* | \, v \in V\}$ as a copy of V, and for sets $X \subset V$, use X^* to denote the set of vertices in V^* corresponding to X. Then, in order to find a 1-factor in $D_{5-in,5-out}$, define an auxiliary bipartite graph $\mathbf{BIP}(V, V^*)$ over the vertex set $V \cup V^*$ whose edges are given as following: for every (directed) edge (u, v) of $D_{5-in,5-out}$, add the (undirected) edge (u, v^*) to $\mathbf{BIP}$. Note that perfect matchings of $\mathbf{BIP}$ has a natural one-to-one correspondence with 1-factors of $D_{5-in,5-out}$. Moreover, the edge distribution of $\mathbf{BIP}$ easily follows from the edge distribution of $D_{5-in,5-out}$. We will say that $D_{5-in,5-out}$ is the *underlying directed graph* of $\mathbf{BIP}$. A permutation σ of V^* acts on $\mathbf{BIP}$ to construct another bipartite graph which has edges $(v, \sigma(w^*))$ for all edges (v, w^*) in $\mathbf{BIP}$.

Our plan is to find a perfect matching which is (almost) a uniform random permutation, and show that this permutation has at most $O(\log n)$ cycles (if it were a uniform random permutation, then this is a well-known result, see, e.g., [12]). Since our distribution is not a uniform distribution, we will rely on the following lemma. Its proof is rather technical, and to avoid distraction, it will be given in the end of this subsection.

Lemma 5.3. *Let X be subset of V. Assume that whp, (i)* $\mathbf{BIP}$ *contains a perfect matching, (ii) every cycle of the underlying directed graph $D_{5-in,5-out}$ contains at least one element from X, and (iii) the edge distribution of* $\mathbf{BIP}$ *is invariant under arbitrary permutations of X^*. Then whp, there exists a perfect matching which when considered as a permutation contains at most $2 \log n$ cycles.*

The next set of lemmas establish the fact that $\mathbf{BIP}$ satisfies all the conditions we need in order to apply Lemma 5.3. First we prove that $\mathbf{BIP}$ contains a perfect matching. We use the following version of the well-known Hall's theorem (see, e.g., [10]).

Theorem 5.4. *Let Γ be a bipartite graph with vertex set $X \cup Y$ and $|X| = |Y| = n$. If for all $X' \subset X$ of size $|X'| \leq n/2$, $|N(X')| \geq |X'|$ and for all $Y' \subset Y$ of size $|Y'| \leq n/2$, $|N(Y')| \geq |Y'|$, then G contains a perfect matching.*

Lemma 5.5. *The graph* $\mathbf{BIP}$ *contains a perfect matching whp.*

Proof. We will verify Hall's condition for the graph $\mathbf{BIP}$ to prove the existence of a perfect matching. Recall that $\mathbf{BIP}$ is a bipartite graph over the vertex set $V \cup V^*$.

Let us show that every set $D \subset V$ of size $|D| \leq n/2$ satisfies $|N(D)| \geq |D|$. This will be done in two steps. First, if $D \subset B_2$, then this follows from the fact that $OUT(v)$ are distinct sets for all $v \in B_2$, (if they were not distinct, then there will be two restricted vertices which are at distance 2 away, and it violates property (v) of Definition 4.7). Second, we prove that for $D \subset V \setminus B_2$,

$$| \, N(D) \cap (V^* \setminus N(B_2)) \, | \geq |D|.$$

It is easy to see that the above two facts prove our claim.

Let $D \subset V \setminus B_2$ be a set of size at most $k \leq n/2$. The inequality $|N(D) \cap (V^* \setminus N(B_2))| < |D|$ can happen only if there exists a set $N^* \subset V^* \setminus N(B_2)$ such that $|N^*| < k$, and for all $v \in D$ all the vertices of $OUT(v)$ belong to $N^* \cup N(B_2)$. Since $D \subset V \setminus B_2$, every vertex in D has 5 random neighbors distributed uniformly over some set of size $(1-o(1))n$, and thus the probability of the above event happening is at most,

$$k\binom{n}{k}^2 \left(\frac{|N(B_2)| + |N^*|}{(1-o(1))n}\right)^{5k} \leq \left(\frac{e^2n^2(\log^{13} n + k)^5}{k^2 \cdot (1-o(1))n^5}\right)^k \leq \left(\frac{9(\log^{13} n + k)^5)}{k^2n^3}\right)^k.$$

For the range $9n/20 \leq k \leq n/2$, we will use the following bound

$$k\binom{n}{k}^2 \left(\frac{\log^{13} n + k}{(1-o(1))n}\right)^{5k} \leq 2^{2n}\left(\frac{1+o(1)}{2}\right)^{9n/4} \leq 2^{-n/5}.$$

Summing over all choices of k we get,

$$\begin{aligned}
&\sum_{k=1}^{n/2} k\binom{n}{k}^2 \left(\frac{\log^{13} n + k}{(1-o(1))n}\right)^{5k} \\
&\leq \sum_{k=1}^{\log^{14} n} \left(\frac{9(\log^{13} n + k)^5}{k^2n^3}\right)^k + \sum_{k=\log^{14} n}^{9n/20} \left(\frac{9(\log^{13} n + k)^5}{k^2n^3}\right)^k + \sum_{k=9n/20}^{n/2} 2^{-n/5} \\
&\leq \sum_{k=1}^{\log^{14} n} \left(\frac{10\log^{70} n}{n^3}\right)^k + \sum_{k=\log^{14} n}^{9n/20} \left(\frac{10k^3}{n^3}\right)^k + o(1) = o(1).
\end{aligned}$$

This finishes the proof that *whp* $|N(D)| \geq |D|$ for all $D \subset V$ of size at most $n/2$. Similarly, for sets $D^* \subset V^*$ of size $|D^*| \leq n/2$, using the sets $IN(v)$ instead of $OUT(v)$ we can show that *whp* $|N(D^*)| \geq |D^*|$ in **BIP**. □

For restricted vertices v, the sets $OUT(v)$ and $IN(v)$ are of size 1 and are already fixed since we fixed the configuration. Thus the edge corresponding to theses vertices will be in **BIP**. Let

$$\hat{A} = A \setminus (\cup_{v \in B_2} OUT(v)),$$

and let $\hat{A}^*$ be the corresponding set inside V^* (note that $\hat{A}$ and $\hat{A}^*$ are fixed sets). This set will be our set X when applying Lemma 5.3. We next prove that every cycle of $D_{5-in,5-out}$ contains vertices of $\hat{A}$.

Lemma 5.6. *Whp, every cycle C of $D_{5-in,5-out}$ contains at least $\lceil \frac{9}{10}|C| \rceil$ vertices of $\hat{A}$.*

Proof. Recall that by Proposition 5.1, for vertices $v \in V \setminus B_2$, the set $OUT(v)$ and $IN(v)$ are uniformly distributed over $V \setminus B_2$, or A. Therefore, for a vertex $w \in B_2$, the only out-neighbor of w is $OUT(w)$, and the only in-neighbor is $IN(w)$ (note that they are both fixed since we fixed the configuration). Also note that,

$$|V \setminus \hat{A}| \leq |V \setminus A| + |B_2| \leq |B_1| + 2|B_2| \leq \frac{(\log\log n)^{12}}{\log^2 n}n + 2\log^{13} n \leq \frac{n}{\log n}.$$

We want to show that in the graph $D_{5-in,5-out}$, *whp* every cycle of length k has at most $k/10$ points from $V \setminus \hat{A}$, for all $k = 1, \ldots, n$. Let us compute the expected number of cycles for which this condition fails and show that it is $o(1)$. First choose k vertices $v_1, v_2, \cdots, v_k$ (with order) and assume that a of them are in B_2. Then since we already know the (unique) out-neighbor and in-neighbor for vertices in B_2, for the vertices $v_1, \cdots, v_k$ to form a cycle in that order, we must fix $3a$ positions (a for the vertices in B_2, and $2a$ for the in-, and out-neighbors of them by property (v) of Definition 4.7). Assume that among the remaining $k - 3a$ vertices, ℓ vertices belong to $V \setminus (\hat{A} \cup B_2)$. Then for there to be at least $\lceil k/10 \rceil$ vertices among $v_1, \cdots, v_k$ not in $\hat{A}$, we must have $3a + \ell \geq \lceil k/10 \rceil$. There are at most 3^k ways to assign one of the three types $\hat{A}$, B_2, and $V \setminus (\hat{A} \cup B_2)$ to each of $v_1, \cdots, v_k$. Therefore the number of ways to choose k vertices as above is at most

$$3^k \cdot n^{k-\ell-3a} |V \setminus \hat{A}|^\ell |B_2|^a \leq 3^k \cdot n^{k-\ell-3a} \left(\frac{n}{\log n} \right)^\ell (\log^{13} n)^a$$

There are $k - 2a$ random edges which has to be present in order to make the above k vertices into a cycle. For all $i \leq k-1$, the pair (v_i, v_{i+1}) can become an edge either by $v_{i+1} \in OUT(v_i)$ or $v_i \in IN(v_{i+1})$ (and also for the pair (v_1, v_k)). There are two ways to choose where the edge $\{v_i, v_{i+1}\}$ comes from, and if both v_i and v_{i+1} are not in B_2, then $\{v_i, v_{i+1}\}$ will become an edge with probability at most $\frac{5}{(1-o(1))n}$. Therefore the probability of a fixed $v_1, \cdots, v_k$ chosen as above being a cycle is at most $2^{k-2a} \left(\frac{5}{(1-o(1))n} \right)^{k-2a}$, and the expected number of such cycles is at most

$$\begin{aligned}
&2^{k-2a} \left(\frac{5}{(1-o(1))n} \right)^{k-2a} \cdot 3^k \cdot n^{k-\ell-3a} \left(\frac{n}{\log n} \right)^\ell (\log^{13} n)^a \\
\leq &\left(\frac{\log^{13} n}{n} \right)^a \cdot \left(\frac{1}{\log n} \right)^\ell \cdot (30 + o(1))^k \\
\leq &\left(\frac{\log^{13} n}{n} \right)^a \cdot \left(\frac{1}{\log n} \right)^{\lceil k/10 \rceil - 3a} \cdot (30 + o(1))^k \leq \left(\frac{\log^{16} n}{n} \right)^a \cdot \left(\frac{40}{(\log n)^{1/10}} \right)^k .
\end{aligned}$$

where we used $3a + \ell \geq \lceil k/10 \rceil$ for the second inequality. Sum this over $0 \leq \ell \leq k$ and $0 \leq a \leq k$ and we get

$$\sum_{k=1}^n \sum_{\ell=0}^k \sum_{a=0}^k \left(\frac{\log^{16} n}{n} \right)^a \cdot \left(\frac{40}{(\log n)^{1/10}} \right)^k = O\left(\sum_{k=1}^n (k+1) \left(\frac{40}{(\log n)^{1/10}} \right)^k \right) = o(1),$$

which proves our lemma.

□

The following simple observation is the last ingredient of our proof.

Lemma 5.7. *The distribution of* **BIP** *is invariant under the action of an arbitrary permutation of* $\hat{A}^*$.

Proof. This lemma follows from the following three facts about the distribution of $D_{5-in,5-out}$. First, all the saturated vertices have the same distribution of IN. Second, for the vertices $v \in V \setminus B_2$, the distribution of OUT and IN is uniform over a set which contains all the saturated vertices (for some vertices it is $V \setminus B_2$, and for others it is A). Third, for the vertices $v \in B_2$, the set $OUT(v)$ lies outside $\hat{A}$ by definition. Therefore, the action of an arbitrary permutation of $\hat{A}^*$ does not affect the distribution of **BIP**. $\square$

Note that here it is important that we fixed the configuration beforehand, as otherwise the set $\hat{A}^*$ will vary, and a statement such as Lemma 5.7 will not make sense.

By combining Lemmas 5.3, 5.5, 5.6, and 5.7, we obtain Proposition 5.2.

Proof of Proposition 5.2. Lemmas 5.5, 5.6, and 5.7 show that the graph **BIP** has all the properties required for the application of Lemma 5.3 (we use $X = \hat{A}$). Thus we know that *whp*, $D_{5-in,5-out}$ has a 1-factor containing at most $2 \log n$ cycles, and in which at least $9/10$ proportion of each cycle are saturated vertices (second property by Lemma 5.6). $\square$

We conclude this subsection with the proof of Lemma 5.3.

Proof of Lemma 5.3. For simplicity of notation, we use the notation $\mathcal{B}$ for the random bipartite graph **BIP**. Note that both a 1-factor over the vertex set V and a perfect matching of (V, V^*), can be considered as a permutation of V. Throughout this proof we will not distinguish between these interpretations and treat 1-factors and perfect matchings also as permutations.

First, let f be an arbitrary function which for every bipartite graph, outputs one fixed perfect matching in it. Then, given a bipartite graph Γ over the vertex set $V \cup V^*$, let Φ be the random variable $\Phi(\Gamma) := \tau^{-1} f(\tau\Gamma)$, where τ is a permutation of the vertices $\hat{A}^*$ chosen uniformly at random. Since the distribution of $\mathcal{B}$ and the distribution of $\tau\mathcal{B}$ are the same by condition (iii), for an arbitrary permutation σ of $\hat{A}^*$, Φ has the following property,

$$\begin{aligned}\mathbb{P}(\Phi(\mathcal{B}) = \phi) = \mathbb{P}(\tau^{-1} f(\tau\mathcal{B}) = \phi) &\overset{(*)}{=} \mathbb{P}((\tau\sigma)^{-1} f(\tau\sigma\mathcal{B}) = \phi) \\ = \mathbb{P}(\tau^{-1} f(\tau\sigma\mathcal{B}) = \sigma\phi) &\overset{(*)}{=} \mathbb{P}(\tau^{-1} f(\tau\mathcal{B}) = \sigma\phi) = \mathbb{P}(\Phi(\mathcal{B}) = \sigma\phi). \end{aligned} \tag{9}$$

In the $(*)$ steps, we used (iii), and the fact that if τ is a uniform random permutation of $\hat{A}^*$, then so is $\tau\sigma$, and therefore, $\mathcal{B}, \tau\mathcal{B}$, and $\tau\sigma\mathcal{B}$ all have identical distribution.

Define a map Π from the 1-factors over the vertex set V to the 1-factors over the vertex set $\hat{A}$ obtained by removing all the vertices that belong to $V \setminus \hat{A}$ from every cycle. For example, a cycle of the form $(x_1 x_2 y_1 y_2 x_3 y_3 x_4)$ will become the cycle $(x_1 x_2 x_3 x_4)$ when mapped by Π (where $x_1, \ldots, x_4 \in \hat{A}$, and $y_1, y_2, y_3 \in V \setminus \hat{A}$). Note that if all the original 1-factors contained at least one element from $\hat{A}$, then the total number of cycles does not change after applying the map Π. This observation combined with condition (ii) implies that it suffices to obtain a bound on the number of cycles after applying Π.

Let σ, ρ be permutations of the vertex set $\hat{A}^*$. We claim that for every 1-factor ϕ of the vertex set V, the equality $\sigma \cdot \Pi(\phi) = \Pi(\sigma \cdot \phi)$ holds. This claim together with (9) gives us,

$$\begin{aligned}\mathbb{P}(\Pi(\Phi(\mathcal{B})) = \rho) = \mathbb{P}(\Phi(\mathcal{B}) \in \Pi^{-1}(\rho)) &\overset{(9)}{=} \mathbb{P}(\sigma\Phi(\mathcal{B}) \in \Pi^{-1}(\rho)) = \mathbb{P}(\Pi(\sigma\Phi(\mathcal{B})) = \rho) \\ &= \mathbb{P}(\sigma \cdot \Pi(\Phi(\mathcal{B})) = \rho) = \mathbb{P}(\Pi(\Phi(\mathcal{B})) = \sigma^{-1}\rho).\end{aligned}$$

Since σ and ρ were an arbitrary permutation of the vertex set $\hat{A}$, we can conclude that conditioned on there existing a perfect matching, $\Pi(\Phi(\mathcal{B}))$ has a uniform distribution over the permutations of $\hat{A}$. It is a well-known fact (see, e.g., [12]) that a uniformly random permutation over a set of size n has *whp* at most $2 \log n$ cycles. Since $\mathcal{B}$ *whp* contains a perfect matching by condition (i), it remains to verify the equality $\sigma \cdot \Pi(\phi) = \Pi(\sigma \cdot \phi)$. Thus we conclude the proof by proving this claim.

For a vertex $x \in \hat{A}$, assume that the cycle of ϕ which contains x is of the form $(\cdots xy_1y_2 \cdots y_k x_+ \cdots)$ $(k \geq 0)$ for $y_1, \ldots, y_k \in V \backslash \hat{A}$. Then by definition $\Pi(\phi)(x) = x_+$, and thus $(\sigma \cdot \Pi(\phi))(x) = \sigma(x_+)$. On the other hand, since σ only permutes $\hat{A}$ and fixes every other element of V, we have $(\sigma \cdot \phi)(x) = \sigma(y_1) = y_1$, and $(\sigma \cdot \phi)(y_i) = y_{i+1}$ for all $i \leq k-1$, and $(\sigma \cdot \phi)(y_k) = \sigma(x_+)$. Therefore the cycle in $\sigma \cdot \phi$ which contains x will be of the form $(\cdots xy_1y_2 \cdots y_k \sigma(x_+) \cdots)$, and then by definition we have $(\Pi(\sigma \cdot \phi))(x) = \sigma(x_+)$. $\square$

5.3 Combining the cycles into a Hamilton cycle

Assume that as in the previous subsection, we started with a fixed typical configuration $\mathbf{c}$, conditioned on the edge process having configuration $\mathbf{c}$, and found a 1-factor of $D_{5-in,5-out}$ by using Proposition 5.2. Since this 1-factor only uses the edges which have been used to construct the graph $D_{5-in,5-out}$, it is independent of the A-A edges in Step II that we did not reveal. Moreover, by the definition of a typical configuration, there are at least $\frac{1}{3} n \log n$ such edges. Note that the algorithm gives a random direction to these edges. So interpret this as receiving $\frac{1}{3} n \log n$ randomly directed A-A edges with repeated edges allowed. Then the problem of finding a directed Hamilton cycle in D_{m_*} can be reduced to the following problem.

Let V be a given set and A be a subset of size $(1 - o(1))n$. Assume that we are given a 1-factor over this vertex set, where at least $9/10$ proportion of each cycle lies in the set A. If we are given $\frac{1}{3} n \log n$ additional A-A edges chosen uniformly at random, can we find a directed Hamilton cycle?

To further simplify the problem, we remove the vertices $V \setminus A$ out of the picture. Given a 1-factor over the vertex set V, mark in red, all the vertices not in A. Pick any red vertex v, and assume that $v_-, v, v_+ \in V$ appear in this order in some cycle of the given 1-factor. If $v_- \neq v_+$, replace the three vertices v_-, v, v_+ by a new vertex v', where v' takes as in-neighbors the in-neighbors of v_-, and as out-neighbors, the out-neighbors of v_+. We call the above process as a *compression* of the three vertices v_-, v, v_+. A crucial property of compression is that every 1-factor of the compressed graph corresponds to a 1-factor in the original graph (with the same number of cycles). Since a directed Hamilton cycle is also a 1-factor, if we can find a Hamilton cycle in the compressed graph, then we can also find one in the original graph.

Now for each $v \in V \setminus A$, compress the three vertices v_-, v, v_+ into a vertex v' and mark it red if and only if either v_- or v_+ is a red vertex. This process always decreases the number of red vertices. Repeat it until there are no red vertices remaining, or $v_- = v_+$ for all red vertices v. As long as there is no red vertex in a cycle of length 2 at any point of the process, the latter will not happen. Consider a cycle whose length was k at the beginning. Since at least $9/10$ proportion of each cycle comes from A and every compression decreases the number of vertices by 2, at any time there will be at least $(8/10)k$ non-red vertices, and at most $(1/10)k$ red vertices remaining in the cycle. Thus if a cycle has a red vertex, then its length will be at least 9, and this prevents length 2 red cycles. So the compressing procedure will be over when all the red vertices disappear. Note that since $|V \setminus A| = |B| = o(n)$, the number of remaining vertices after the compression procedure is over is at least $n - 2|B| = (1 - o(1))n$. As mentioned above, it suffices to find a Hamilton cycle in the graph after the compression process is over.

Another important property of this procedure is related to the additional A-A edges that we are given. Assume that v is the first red vertex that we have compressed, where the vertices v_-, v, v_+ appeared in this order in some 1-factor. Further assume that v_- and v_+ are not red vertices. Then since the new vertex v' obtained from the compression will take as out-neighbors the out-neighbors of v_+, and in-neighbors the in-neighbors of v_-, we may assume that this vertex v' is a vertex in A from the perspective of the new $\frac{1}{3}n \log n$ edges that will be given.

This observation shows that every pair of vertices of the compressed graph has the same probability of being one of the new $\frac{1}{3}n \log n$ edges. Since the number of vertices reduced by $o(n)$, only $o(n \log n)$ of the new edges will be lost because of the compression. Thus *whp* we will be given $(\frac{1}{3} - o(1))n \log n$ new uniform random edges of the compressed graph.

Theorem 5.8. *For a typical configuration* $\mathbf{c}$*, conditioned on the random edge process having configuration* $\mathbf{c}$*, the directed graph* D_{m_*} *whp contains a Hamilton cycle.*

Proof. By Proposition 5.2, there exists *whp* a perfect matching of **BIP** which corresponds to a 1-factor in D_{m_*} consisting of at most $2 \log n$ cycles. Also, at least $9/10$ proportion of the vertices in each cycle lies in A. After using the compression argument which has been discussed above, we may assume that we are given a 1-factor over some vertex set of size $(1 - o(1))n$. Moreover, the random edge process contains at least $(\frac{1}{3} - o(1))n \log n$ additional random directed edges (distributed uniformly over that set). By Theorem 3.1 with L being the whole vertex set, we can conclude that *whp* the compressed graph contains a directed Hamilton cycle, and this in turn implies that D_{m_*} contains a directed Hamilton cycle. $\square$

Corollary 5.9. *The directed graph* D_{m_*} *whp contains a Hamilton cycle.*

Proof. Let $\mathbf{e}$ be a random edge process. Let $D = D_{m_*}(\mathbf{e})$ and $\mathcal{HAM}$ be the collection of directed graphs that contain a directed Hamilton cycle. For a configuration $\mathbf{c}$, denote by $\mathbf{e} \in \mathbf{c}$, the event that $\mathbf{e}$ has configuration $\mathbf{c}$. If $\mathbf{e} \in \mathbf{c}$ for some typical configuration $\mathbf{c}$, then we say that $\mathbf{e}$ is typical.

By Theorem 5.8, we know that for any typical configuration $\mathbf{c}$, $\mathbb{P}(D \notin \mathcal{HAM} | \mathbf{e} \in \mathbf{c}) = o(1)$, from which we know that $\mathbb{P}(\{D \notin \mathcal{HAM}\} \cap \{\mathbf{e} \text{ is typical}\}) = o(1)$. On the other hand, by Lemma 4.8 we know that the probability of an edge process having a non-typical configuration is $o(1)$. Therefore *whp*, the directed graph D is Hamiltonian $\square$

6 Going back to the original process

Recall that the distribution of the random edge process is slightly different from that of the random graph process since it allows repeated edges and loops. In fact, one can show that at time m_*, the edge process *whp* contains at least $\Omega(\log^2 n)$ repeated edges. Therefore, we cannot simply condition on the event that the edge process does not contain any repeated edges or loops to obtain our main theorem for random graph processes. Our next theorem shows that there exists an on-line algorithm **OrientPrime** which successfully orients the edges of the random graph process.

Theorem 6.1. *There exists a randomized on-line algorithm* **OrientPrime** *which orients the edges of the random graph process, so that the resulting directed graph is Hamiltonian whp at the time at which the underlying graph has minimum degree 2.*

The algorithm **OrientPrime** will mainly follow **Orient** but with a slight modification. Assume that we are given a random graph process (call it the underlying process). Using this random graph process, we want to construct an auxiliary process whose distribution is identical to the random edge process. Let $t = 1$ at the beginning and a_t be the number of distinct edges up to time t in our auxiliary process (disregarding loops). Thus $a_1 = 0$. At time t, with probability $(2a_t + n)/n^2$ we will produce a redundant edge, and with probability $1 - (2a_t + n)/n^2$, we will receive an edge from the underlying random graph process. Once we decided to produce a redundant edge, with probability $2a_t/(2a_t + n)$ choose uniformly at random an edge out of the a_t edges that already appeared, and with probability $n/(2a_t + n)$ choose uniformly at random a loop. Let e_t be the edge produced at time t (it is either a redundant edge, or an edge from the underlying process), and choose its first vertex and second vertex uniformly at random. One can easily check that the process $(e_1, e_2, \cdots,)$ has the same distribution as the random edge process.

In the algorithm **OrientPrime**, we feed this new auxiliary process into the algorithm **Orient** and orient the edges accordingly. Since the distribution of the auxiliary process is the same as that of the random edge process, **Orient** will give an orientation which *whp* contains a directed Hamilton cycle. However, what we seek for is a Hamilton cycle with no redundant edge. Thus in the edge process, whenever we see a redundant edge that is a repeated edge (not a loop), color it by blue. In order to show that **OrientPrime** gives a Hamiltonian graph *whp*, it suffices to show that we can find a Hamilton cycle in D_{m_*} which does not contain a blue edge (note that loops cannot be used in constructing a Hamilton cycle). We first state two useful facts.

Claim 6.2. *Whp, there are no blue edges incident to B used in constructing $D_{5-in,5-out}$.*

Proof. The expected number of blue edges incident to B in Step I used in constructing $D_{5-in,5-out}$ can be computed by choosing two vertices v and w and then computing the probability that $v \in B$, and (v, w) or (w, v) together appears twice among Step I edges. The probability that v appears as a first vertex exactly i times is $\binom{n \log\log n}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{n\log\log n - i}$. Condition on the event that v appeared i times as a first vertex for some $i < 12$ (and also reveal the i positions in which v appeared). We then compute the probability that some two Step I edges are (v, w) or (w, v). There are three events that we need to consider. First is the event that (v, w) appears twice, whose probability is $\binom{i}{2}\left(\frac{1}{n}\right)^2$. Second is the event that (v, w) appears once and (w, v) appears once, whose probability is at most $\binom{n\log\log n}{1}\frac{1}{n(n-1)} \cdot \binom{i}{1}\frac{1}{n}$. Third is the event that (w, v) appears twice, whose probability is at most $\binom{n\log\log n}{2}\left(\frac{1}{n(n-1)}\right)^2$. Combining everything, we see that the expected number of Step I blue edges incident to B is at most,

$$n^2 \cdot \sum_{i=0}^{11} \binom{n\log\log n}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{n\log\log n - i} \times$$
$$\left(\binom{i}{2}\left(\frac{1}{n}\right)^2 + \binom{n\log\log n}{1}\left(\frac{1}{n(n-1)}\right)\binom{i}{1}\frac{1}{n} + \binom{n\log\log n}{2}\left(\frac{1}{n(n-1)}\right)^2\right).$$

The main term comes from $i = 11$, and the third term in the final bracket. Consequently, we can bound the expectation by

$$(1 + o(1)) \cdot n^2 \cdot \binom{n\log\log n}{11}\left(\frac{1}{n}\right)^{11}\left(1 - \frac{1}{n}\right)^{n\log\log n - 11} \cdot \binom{n\log\log n}{2}\left(\frac{1}{n(n-1)}\right)^2 = o(1).$$

We then would like to compute the expected number of blue edges incident to B in Step 2 used in constructing $D_{5-in,5-out}$. Condition on the first vertices of the Step I edges so that we can determine the sets A and B. By Claim 4.3, we may condition on the event $|B| = O(\frac{(\log\log n)^{12}}{\log^2 n})$. Fix a vertex $v \in B$, and expose all appearances of v in Step II, and note that only the first 10 appearances are relevant. By Claim 4.4, it suffices to bound the probability of the event that there exists a vertex $w \in A$ such that (v, w) or (w, v) appears twice among the at most 24 Step I edges where v or w are the first vertices, and the at most 10 Step II edges which we know is going to be used to construct the OUT and IN of the vertex v. Therefore the expectation is

$$|B| \cdot n \cdot \left(\frac{34}{n}\right)^2 = O\left(\frac{(\log\log n)^{12}}{\log^2 n} n^2\right) \cdot \left(\frac{34}{n}\right)^2 = o(1).$$

$\square$

Claim 6.3. *Whp, there are at most* $\log n$ *blue edges used in constructing* $D_{5-in,5-out}$.

Proof. By Claim 6.2, we know that whp, all the blue edges used in constructing $D_{5-in,5-out}$ are incident to A. Therefore it suffices to show that there are at most $\log n$ blue edges among the Step I edges. The expected number of such edges can be computed by choosing two vertices v, w, and computing the probability that (v, w) or (w, v) appears twice. Thus is at most

$$n^2 \cdot \binom{n\log\log n}{2} \left(\frac{2}{n^2}\right)^2 = o(\log n).$$

Consequently, by Markov's inequality, we can derive the conclusion. $\square$

Claim 6.4. *Whp, each vertex is incident to at most one blue edge.*

Proof. It suffices to show that there does not exist three distinct vertices v, w_1, w_2 such that both $\{v, w_1\}$ and $\{v, w_2\}$ appear at least twice. The probability of this event is at most

$$\binom{n}{3}\binom{m_2}{4} \cdot \binom{4}{2} \left(\frac{2}{n^2}\right)^4 = o(1).$$

$\square$

Now assume that we found a 1-factor as in Section 5.2. By Claim 6.3, *whp*, it contains at most $\log n$ blue edges. Then after performing the compression process given in the beginning of Section 5.3, by Claim 6.2, the number of blue edges remains the same as before. Therefore, if we can find a Hamilton cycle in the compressed graph which does not use any of the blue edges, then the original graph will also have a Hamilton cycle with no blue edges. Thus our goal now is to combine the cycles into a Hamilton cycle without any blue edges, by using the non-revealed A-A edges.

In order to do this, we provide a proof of a slightly stronger form of Theorem 3.1 for $L = V$. In fact, it can be seen that when combined with the compression argument, this special case of the theorem implies the theorem for general L. Note that we have at least $\frac{n\log n}{3}$ non-revealed A-A edges remaining after finding the 1-factor described in the previous paragraph. Note that these edges cannot create more blue edges in the 1-factor we previously found, since all the A-A edges used so far appears earlier in the process than these non-revealed edges. We will find a Hamilton cycle in two more phases.

The strategy of our proof comes from that of Frieze [14]. In the first phase, given a 1-factor consisting of at most $O(\log n)$ cycles, we use the first half of the remaining non-revealed A-A edges to combine some of the cycles into a cycle of length $n - o(n)$. In this phase, we repeatedly combine two cycles of the 1-factor until there exists a cycle of length $n - o(n)$.

Lemma 6.5. *Whp, there exists a 1-factor consisting of $O(\log n)$ cycles, one of which is of length $n - o(n)$. Moreover, this 1-factor contains at most $O(\log n)$ blue edges.*

Proof. Condition on the conclusion of Claim 6.3. Then we are given a 1-factor consisting of at most $c \log n$ cycles and containing at most $\log n$ blue edges. Our goal is to modify this 1-factor into a 1-factor satisfying the properties as in the statement. Consider the non-revealed random A-A edges we are given. Since we will use only the first half of these edges, we have at least $\frac{n \log n}{6}$ random A-A edges given uniformly among all choices. Let E_N be these edges. Partition E_N as $E_0 \cup E_1 \cup \cdots \cup E_{c \log n}$, where E_0 is the first half of edges, E_1 is the next $\frac{1}{2c \log n}$ proportion of edges, E_2 is the next $\frac{1}{2c \log n}$ proportion of edges, and so on. Thus $|E_0| = \frac{1}{2}|E_N|$ and $|E_1| = \cdots = |E_{c \log n}| = \frac{1}{2c \log n}|E_N|$. Since $|E_0| \geq \frac{n \log n}{12}$, by applying Chernoff's inequality and taking the union bound, we can see that *whp*, for every set of vertices X of size at least $|X| \geq \frac{n}{\log^{1/2} n}$, there exists at least $\frac{1}{2}|X||V \setminus X| \frac{\log n}{12n} \geq \frac{n \log^{1/2} n}{48}$ edges of E_0 between X and $V \setminus X$. Condition on this event.

Assume that the 1-factor currently does not contain a cycle of length at least $n - \frac{2n}{\log^{1/2} n}$. Then we can partition the cycles into two sets so that the number of vertices in the cycles belonging to each part is between $\frac{n}{\log^{1/2} n}$ and $n - \frac{n}{\log^{1/2} n}$. Thus by the observation above, there exist at least $\frac{n \log^{1/2} n}{48}$ edges of E_0 between the two parts. Let (v, w) be one such edge. Let v^+ be the vertex that succeeds v in the cycle of the 1-factor that contains v, and let w^- be the vertex that precedes w in the cycle of the 1-factor that contains w. If $(w^-, v^+) \in E_1$, then the cycle containing v and the cycle containing w can be combined into one cycle (see Figure 1). Therefore, each edge in E_0 gives rise to some pair e for which if $e \in E_1$, then some two cycles of the current 1-factor can be combined into another cycle. The probability of no such edge being present in E_1 is at most

$$\left(1 - \frac{1}{n^2} \cdot \frac{n \log^{1/2} n}{48}\right)^{|E_1|} \leq e^{-\left(\log^{1/2} n/(48n)\right) \cdot \left(|E_N|/(2c \log n)\right)} \leq e^{-\Omega(\log^{1/2} n)}.$$

Therefore with probability $1 - e^{-\Omega(\log^{1/2} n)}$, we can find an edge in E_0 and an edge in E_1 which together will reduce the total number of cycles in the 1-factor by one.

We can repeat the above using E_i instead of E_1 in the i-th step. Since the total number of cycles in the initial 1-factor is at most $c \log n$, the process must terminate before we run out of edges. Therefore at some step, we must have found a 1-factor that has at most $O(\log n)$ cycles, and contains a cycle of length $n - o(n)$. It suffices to check that the estimate on the number of blue edges hold. Indeed, every time we combine two cycles, we use two additional edges which are not in the 1-factor, and therefore by the time we are done, we would have added $O(\log n)$ edges to the initial 1-factor. Therefore even if all these edges were blue edges, we have $O(\log n)$ blue edges in the 1-factor in the end. □

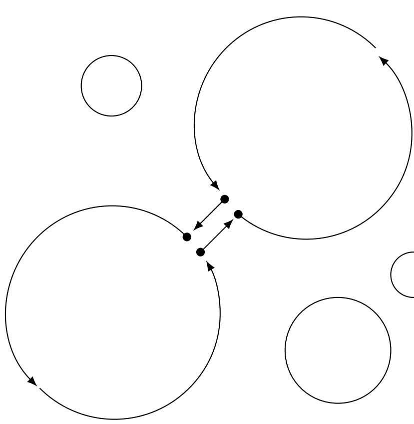
Consider a 1-factor given by the previous lemma. In the second phase, we use the other half of the remaining new random edges to prove that the long cycle we just found, can "absorb" the remaining cycles. Let $P = (v_0, \cdots, v_\ell)$ be a path of a digraph. If there exist two edges (v_ℓ, v_{i+1}) and (v_i, v_j) for

$1 \le i < \ell$ and $i+1 < j \le \ell$, then we can *rotate* the path P using v_i and v_{j-1} as *breaking points* to obtain a new path $(v_0, v_1, \cdots, v_i, v_j, v_{j+1}, \cdots, v_\ell, v_{i+1}, v_{i+2}, \cdots, v_{j-1})$ (see Figure 1). We call v_i the *intermediate point* of this rotation. Note that if the graph contains the edge (v_{j-1}, v_0), then one can *close* the path into a cycle. Our strategy is to repeatedly rotate the given path until one can find such an edge and close the path (see Figure 1).

Further note that the path obtained from P by rotating it once as above can be described as following. Let P_1, P_2, P_3 be subpaths of P obtained by removing the edges (v_i, v_{i+1}) and (v_{j-1}, v_j). Then there exists a permutation π of the set $[3]$ such that the new path is the path obtained by concatenating $P_{\pi(1)}, P_{\pi(2)}, P_{\pi(3)}$ (in order). More generally, assume that we rotate the path P in total s times by using distinct breaking points $v_{a_1}, v_{a_2}, \cdots, v_{a_{2s}}$. Let $P_1, \cdots, P_{2s+1}$ be the subpaths of P obtained by removing the edges (v_{a_j}, v_{a_j+1}) for $1 \le j \le 2s$. Then there exists a permutation σ of the set $[2s+1]$ such that the path we have in the end is the path obtained by concatenating $P_{\sigma(1)}, P_{\sigma(2)}, \cdots, P_{\sigma(2s+1)}$. We will use this fact later. Note that it is crucial to have distinct breaking points here.

After finding a 1-factor described in Lemma 6.5, there are at least $\frac{n \log n}{6}$ non-revealed A-A edges that we can use. Let E_L be the later $\frac{n \log n}{6}$ of these edges, and reveal all the non-revealed edges not in E_L. Note that there exists a positive constant C such that *whp*, the graph induced by the revealed edges before beginning this phase has maximum degree at most $C \log n$ (it follows from Chernoff's inequality and union bound). Condition on this event.

We will use the remaining edges E_L in a slightly different way from how we did in the previous phase since in this phase, it will be more important to know if some certain edge is present among the non-revealed edges. For an ordered pair of vertices $e = (x, y)$, let the *flip* of e be $r(e) = (y, x)$ (similarly define a flip of some set of pairs). Fix some pair $e = (x, y)$, and suppose that we are interested in knowing whether $e \in E_L$ holds or not, and if $e \in E_L$, then whether it is a blue edge or not. Thus for each of the non-revealed edge in E_L, ask if it is e or $r(e)$. Since we know how many times e and $r(e)$ appeared among the already revealed edges, in the end, we not only know if $e \in E_L$, but also know if if it is a blue edge or not. We call this procedure as *exposing* the pair e, and say that e has been exposed. Note that the process of exposing the pair e is symmetric in the sense that even if we are looking only for the edge e we seek for the existence of $r(e)$ as well. This is because we would



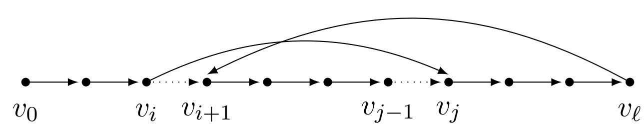


Figure 1: Combining two cycles, and rotating a path.

like to determine whether e is blue or not at the same time. We can similarly define the procedure of exposing a set of pairs, instead of a single pair. We would like to carefully expose the edges in order to construct a Hamilton cycle without blue edges.

Note that the expected number of times that e or $r(e)$ appears in E_L is $\frac{2}{n^2} \cdot \frac{n \log n}{6} = \frac{\log n}{3n}$. Thus if S is the set of exposed pairs at some point, we say that the *outcome is typical* if the number of times that a pair belonging to S appears in E_L is at most $\frac{|S| \log n}{n}$ (which is three times its expected value). While exposing sets of pairs, we will maintain the outcome to be typical, since we would like to know that there are enough non-revealed pairs remaining in E_L. For a set X of vertices, let $Q(X)$ be the set of ordered pairs (x_1, x_2) such that $x_1 \in X$ or $x_2 \in X$.

Lemma 6.6. *Let X and Y be sets of vertices of size at most $\frac{n}{32}$. Assume that the set of exposed pairs so far is a subset of $Q(X)$ and the outcome is typical. Further assume that a path P from v_0 to v_ℓ of length $\ell = n - o(n)$ is given for some $v_0, v_\ell \notin X \cup Y$.*

Then there exists a set $Z \subset V(P)$ disjoint from Y of size at most $|Z| \le \frac{n}{\log n \cdot \log \log n}$ such that with probability at least $1 - o((\log n)^{-1})$, by further exposing only pairs that intersect Z (thus a subset of $Q(Z)$), one can find a cycle over the vertices of P. Furthermore, the outcome of exposing these pairs is typical and no new blue edges are added (thus the set of blue edges in the cycle is a subset of the set of blue edges in P).

Informally, Y is the set of 'reserved' vertices which we would like to keep non-exposed for later usage. The lemma asserts that we can close the given path into a cycle by further exposing pairs that intersect some set Z which is disjoint from Y and has relatively small cardinality.

Proof. Denote the path as $P = (v_0, v_1, \cdots, v_\ell)$. For a subset of vertices $A = \{v_{a_1}, v_{a_2}, \cdots, v_{a_t}\}$, define $A^- = \{v_{a_1-1}, v_{a_2-1}, \cdots, v_{a_t-1}\}$ and $A^+ = \{v_{a_1+1}, v_{a_2+1}, \cdots, v_{a_t+1}\}$ (if the index reaches either -1 or $\ell+1$, then we remove the corresponding vertex from the set).

Our strategy can be described as following. We repeatedly rotate the path to obtain endpoints, and in each iteration select a set of vertices and expose only pairs incident to these vertices (call these vertices as the *involved vertices*). Thus a pair consisting of two non-involved vertices will remain non-exposed. The set Z will be the set of involved vertices, and our goal will be to construct a cycle while maintaining Z to be small.

To keep track of the set of vertices that have been involved and the set of endpoints that we obtained, we maintain two sets T_i and S_i for $i \ge 0$, where $T_0 = \{v_\ell\}$ and $S_0 = X$. Informally, T_i will be the set of endpoints that have not yet been involved, and S_i will be the set of involved vertices while obtaining the set T_i. For example, suppose that we performed a rotation as in Figure 1 in the first round. We will later see that in the process, we expose the neighbors of v_ℓ and v_i for this round of rotation to obtain a new endpoint v_{j-1}. Thus we will add the vertices v_ℓ and v_i to S_1 and v_{j-1} to T_1. It is crucial to maintain T_i as a subset of the set of non-involved vertices, since we will need to expose its neighbors in the next round of rotation.

Let $Y_0 = Y \cup \{v_0\}$. Throughout the rotation process, T_i and S_i will satisfy the following properties:

(i) for every $w \in T_i$, there exists a path of length ℓ from v_0 to w whose set of blue edges is a subset of that of P,
(ii) the set of exposed pairs after the i-th step is a subset of $Q(S_i)$,
(iii) all the breaking points used in constructing the paths above belong to $S_i \cup T_i$,

(iv) $|T_i| = \left(\frac{\log n}{500}\right)^{2i}$ and $|S_i \setminus S_{i-1}| \le 2\left(\frac{\log n}{500}\right)^{2i-1} = \frac{1000}{\log n}|T_i|$ (for $i \ge 1$),
(v) $X \cup T_{i-1} \cup S_{i-1} \subset S_i$,
(vi) S_i, T_i, and Y_0 are mutually disjoint, and
(vii) the outcome at each iteration is typical.

Recall that $T_0 = \{v_\ell\}$ and $S_0 = X$, and note that the properties above indeed hold for these sets. Since $S_0 = X$, property (iv) in particular implies that

$$|S_i| \le |X| + \sum_{a=1}^{i} \frac{1000}{\log n}|T_a| \le |X| + \frac{2000}{\log n}|T_i|.$$

Suppose that we completed constructing the sets T_i and S_i for some index i so that $|T_i| \le \frac{n}{(\log n)^2 \log\log n}$. By (iv), we have $|S_i| \le |X| + \frac{2000n}{\log n}|T_i| \le (\frac{1}{32} + o(1))n$ and $i = O(\frac{\log n}{\log\log n})$. We will show how to construct the sets T_{i+1} and S_{i+1} from these sets.

By $|X| \le \frac{n}{32}$, (ii), (iv) and (vii), we know that at any step of the process the number of edges in E_L that remain non-revealed is at least

$$|E_L| - \frac{|Q(S_i)| \cdot \log n}{n} \ge \frac{n \log n}{6} - \frac{2|S_i| n \log n}{n} \ge \frac{n \log n}{12}.$$

Moreover, the number of non-exposed pairs remaining is at least

$$n^2 - |Q(S_i)| \ge n^2 - 2n|S_i| \ge \frac{n^2}{2}.$$

We will make use of the following three claims whose proof will be given later.

Claim 6.7. *Assume that some pairs have been exposed and the outcome is typical. Once we expose the remaining edges, the probability that there exists a vertex incident to two new blue edges is at most $o((\log n)^{-2})$.*

Claim 6.8. *Assume that some pairs have been exposed and the outcome is typical. Let R be a set of pairs of size $|R| = \Omega(\frac{n}{\log n})$ disjoint to the exposed pairs. Then with probability at least $1 - o((\log n)^{-2})$, the number of times a pair in R appear among the non-revealed edges of E_L is at least $\frac{|R|\log n}{24n}$, and is at most $\frac{|R|\log n}{2n}$.*

Claim 6.9. *Assume that some pairs have been exposed and the outcome is typical. Then with probability at least $1 - o((\log n)^{-2})$, for every disjoint sets A_1, A_2 of vertices satisfying $|A_1| \le \frac{n}{\log n \cdot (\log\log n)^{1/2}}$ and $|A_2| = \frac{|A_1|\log n}{500}$, the number of edges between A_1 and A_2 among the non-revealed edges of E_L is at most $\frac{|A_1|\log n}{100}$.*

For each vertex $w \in T_i$, there exists a path P_w of length ℓ from v_0 to w satisfying (i). Let $P_{w,1}$ be the first half and $P_{w,2}$ be the second half of P_w. Let $S_{i+1,0} = S_i \cup T_i$, and $N = S_{i+1,0} \cup S^{-}_{i+1,0} \cup Y_0$ and

$$Q_1 = \{(w, x^+) : w \in T_i, x \in V(P_{w,1}) \setminus N\}.$$

We have $Q_1 \subset Q(T_i)$ and

$$|Q_1| \ge |T_i| \cdot \left(\frac{\ell}{2} - 2|S_i| - 2|T_i| - |Y| - 1\right) \ge \frac{n}{4}|T_i|.$$

By (vi) and the definition of N, the pairs in Q_1 have both of their endpoints not in S_i, thus have not been exposed yet. Now expose the set Q_1. By Claim 6.8, we know that with probability at least $1 - o((\log n)^{-2})$, the outcome is typical, and the number of pairs in Q_1 that appear in E_L is at least

$$\frac{|Q_1| \log n}{24n} \geq \frac{|T_i| \log n}{96}.$$

Condition on this event. Note that if some pair $(w, x^+) \in Q_1$ appears in E_L and is not a blue edge, then x can serve as an intermediate point in our next round of rotation. Since we forced not to use the same breaking point twice by avoiding the set N (see properties (iii) and (v)), if there is a non-blue edge of the form (x, y^+) for some $y \in P_{w,2}$, then we can find a path of length ℓ from v_0 to y satisfying (i) (see Figure 1).

Let

$$S_{i+1,1} = \{x : (w, x^+) \in Q_1 \cap E_L, (w, x^+) \text{ is not blue}\}.$$

By Claim 6.7, with probability at least $1 - o((\log n)^{-2})$, among the edges in $Q_1 \cap E_L$, the number of blue edges is at most $|T_i|$. Condition on this event. Then the number of non-blue edges between T_i and $S^+_{i+1,1}$ is at least $\frac{|T_i|(\log n - 1)}{96} > \frac{|T_i| \log n}{100}$. By Claim 6.9, with probability at least $1 - o((\log n)^{-2})$, we see that $|S_{i+1,1}| \geq \frac{|T_i| \log n}{500}$. Redefine $S_{i+1,1}$ as an arbitrary subset of it of size exactly $\frac{|T_i| \log n}{500}$. Note that $S_{i+1,1} \cap N = \emptyset$. The vertices in $S_{i+1,1}$ will serve as intermediate points of our rotation.

Now let

$$Q_2 = \{(x, y^+) : x \in S_{i+1,1},\ (w, x^+) \in Q_1 \cap E_L,\ \text{and } y \in V(P_{w,2}) \setminus (N \cup S^-_{i+1,1})\},$$

and note that $Q_2 \subset Q(S_{i+1,1})$. Further note that we are subtracting $S^-_{i+1,1}$ from $V(P_{w,2})$ in the above definition. This is to avoid having both a pair and its reverse in the set Q_2. Even though the set $S_{i+1,1}$ was defined as a collection of vertices belonging to $P_{w',1}$ for various choices of w', it can still intersect $P_{w,2}$ for some vertex w, since we are considering different paths for different vertices. Similarly as before, all the pairs in Q_2 are not exposed yet and we have $|Q_2| \geq \frac{n}{4}|S_{i+1,1}|$. Moreover, with probability at least $1 - o((\log n)^{-2})$, the number of pairs in Q_2 that appear in E_L which are not blue edges is at least $\frac{|S_{i+1,1}| \log n}{100}$ and the outcome is typical. Let $T_{i+1,0} = \{y : (x, y^+) \in Q_2 \cap E_L, (x, y^+) \text{ is not blue}\}$. As in above, with probability at least $1 - o((\log n)^{-2})$, we have $|T_{i+1,0}| \geq \frac{|S_{i+1,1}| \log n}{500}$. Moreover, by the observation above, for all the vertices $y \in T_{i+1,0}$, there exists a path of length ℓ from v_0 to y satisfying (i).

Let $T_{i+1} = T_{i+1,0}$ and $S_{i+1} = S_{i+1,0} \cup S_{i+1,1}$. Since $|T_{i+1}| \geq \left(\frac{\log n}{500}\right)^2 |T_i| \geq \left(\frac{\log n}{500}\right)^{2(i+1)}$, we may redefine T_{i+1} as an arbitrary subset of it of size exactly $\left(\frac{\log n}{500}\right)^{2(i+1)}$. In the previous paragraph we saw that (i) holds for T_{i+1}. Property (ii) holds since the set of newly exposed pairs is $Q_1 \cup Q_2 \subset Q(S_{i+1,0} \cup S_{i+1,1}) = Q(S_{i+1})$. Properties (iii), (v), and (vi) can easily be checked to hold. By Claim 6.8, the outcome is typical, and we have (vii). For property (iv), the size of T_{i+1} by definition satisfies the bound, and the size of $S_{i+1} \setminus S_i$ is

$$\begin{aligned} |S_{i+1} \setminus S_i| &\leq |S_{i+1,0} \setminus S_i| + |S_{i+1,1}| \\ &\leq |T_i| + \frac{|T_i| \log n}{500} \leq \left(\frac{\log n}{500}\right)^{2i} \cdot \left(1 + \frac{\log n}{500}\right) \leq 2\left(\frac{\log n}{500}\right)^{2i+1}. \end{aligned}$$

Repeat the above until we reach a set T_t of size $\frac{n}{(\log n)^2 \cdot \log\log n} \leq |T_t| \leq \frac{n}{(500)^2 \log\log n}$. By (iv), we have $t = O(\frac{\log n}{\log\log n})$ and $|S_t| \leq |X| + \frac{n}{125 \log n \cdot \log\log n}$. Redefine T_t as an arbitrary subset of size exactly $\frac{n}{(\log n)^2 \cdot \log\log n}$. Note that the size of S_t does not necessarily decrease, and thus we still have $|S_t| \leq |X| + \frac{n}{125 \log n \cdot \log\log n}$. We will repeat the process above for the final time with the sets S_t and T_t. This will give $|T_{t+1}| = \frac{n}{(500)^2 \log\log n}$ and $|S_{t+1} \setminus S_t| \leq \frac{n}{125 \log n \cdot \log\log n}$, from which it follows that $|S_{t+1}| \leq \frac{2n}{125 \log n \log\log n}$. Let $Q_3 = \{(v_0, z) : z \in T_{t+1}\}$ and expose Q_3 (note that the pairs in Q_3 has not yet been exposed since $(T_{t+1} \cup \{v_0\}) \cap S_{t+1} = \emptyset$, while the set of exposed pairs is $Q(S_{t+1})$). Since $|Q_3| = |T_{t+1}| = \Omega(\frac{n}{\log\log n})$, by Claims 6.7 and 6.8, with probability at least $1 - o((\log n)^{-2})$, we have a pair in Q_3 that appears in E_L as a non-blue edge. This gives a cycle over the vertices of P whose set of blue edges is a subset of that of P.

For the set $Z = (S_{t+1} \cup \{v_0\}) \setminus X$, we see that the set of exposed pairs is a subset of $Q(X \cup Z)$. Furthermore, since Y_0 and S_{t+1} are disjoint and $v_0 \notin Y$, the sets Y and Z are disjoint as well. By $t = O(\frac{\log n}{\log\log n})$, the total number of events involved is $O(\frac{\log n}{\log\log n})$. Since each event hold with probability at least $1 - o((\log n)^{-2})$, by taking the union bound, we obtain our set and cycle as claimed with probability at least $1 - o((\log n)^{-1})$. □

The proofs of Claims 6.7, 6.8, and 6.9 follow.

Proof of Claim 6.7. Let G' be the graph induced by the edges that have been revealed before the final phase (thus all the edges but E_L). It suffices to compute the probability of the following events: (i) there exist $v, w_1, w_2 \in V$ such that both $\{v, w_1\}$ and $\{v, w_2\}$ appears at least twice among the remaining edges, (ii) there exist $v, w_1, w_2 \in V$ such that $\{v, w_1\}$ and $\{v, w_2\}$ were already in G', and both appears at least once among the remaining edges, and (iii) there exist $v, w_1, w_2 \in V$ such that $\{v, w_1\}$ were already in G', appears at least once among the remaining edges, and $\{v, w_2\}$ appears at least twice among the remaining edges.

The probability of the first event happening is at most

$$n^3 \cdot \binom{n \log n/6}{4} \cdot \binom{4}{2} \left(\frac{2}{n^2}\right)^4 = O\left(\frac{(\log n)^4}{n}\right).$$

Recall that we conditioned on the event that each vertex has degree at most $C \log n$ in the graph induced by the edges revealed before this phase. Consequently, the probability of the second event happening is at most

$$n \cdot \binom{C \log n}{2} \cdot \binom{n \log n/6}{2} \binom{2}{1} \cdot \left(\frac{2}{n^2}\right)^2 = O\left(\frac{(\log n)^4}{n}\right),$$

and similarly, the probability of the third event happening is at most

$$n^2 \cdot \binom{C \log n}{1} \cdot \binom{n \log n/6}{3} \binom{3}{1} \cdot \left(\frac{2}{n^2}\right)^3 = O\left(\frac{(\log n)^4}{n}\right).$$

Therefore we have our conclusion. □

Proof of Claim 6.8. Recall that at any time of the process, the number of non-revealed edges in E_L is at least $\frac{n \log n}{12}$. The probability of a single non-revealed edge of E_L being in R is at least $\frac{|R|}{n^2}$. Therefore the expected number of times a pair in R appear among the non-revealed edges is at least,

$$\frac{|R|}{n^2} \cdot \frac{n \log n}{12} = \frac{|R| \log n}{12n}.$$

On the other hand, recall that at any time of the process, the probability that a non-revealed edge of E_L is some fixed pair at most $\frac{2}{n^2}$, since the number of non-exposed pairs is at least $\frac{n^2}{2}$. Therefore the expected number of times a pair in R appear among the non-revealed edges is at most,

$$\frac{2|R|}{n^2} \cdot \frac{n \log n}{6} = \frac{|R| \log n}{3n}.$$

Since $|R| = \Omega(\frac{n}{\log \log n})$, the conclusion follows from Chernoff's inequality and union bound. □

Proof of Claim 6.9. Recall that at any time of the process, the probability that a non-revealed edge of E_L is (v, w) or (w, v) is at most $\frac{4}{n^2}$, since the number of non-exposed pairs is at least $\frac{n^2}{2}$.

Let k be a fixed integer satisfying $k \leq \frac{n}{\log n \cdot \log \log n}$. Let A_1 be a set of vertices of size k and A_2 be a set of vertices of size $\frac{k \log n}{500}$ disjoint from A_1. The number of choices for such sets is at most

$$n^k \binom{n}{k \log n/500} \leq \left(n^{\frac{500}{\log n}} \cdot \frac{500en}{k \log n} \right)^{k \log n/500} \leq \left(\frac{e^{1000} n}{k \log n} \right)^{k \log n/500}.$$

The probability of there being more than $\frac{k \log n}{100}$ edges between A_1 and A_2 can be computing by first choosing $\frac{k \log n}{100}$ pairs between A_1 and A_2, and then computing the probability that they all appear among the remaining edges. Thus is at most

$$\begin{aligned}
&\binom{k^2 \log n/500}{k \log n/100} \cdot \left(\frac{n \log n}{3} \right)^{k \log n/100} \left(\frac{4}{n^2} \right)^{k \log n/100} \\
&\leq \left(\frac{ek}{5} \cdot \frac{n \log n}{3} \cdot \frac{4}{n^2} \right)^{k \log n/100} \\
&\leq \left(\frac{4ek \log n}{15n} \right)^{k \log n/100} \leq \left(\frac{k \log n}{n} \right)^{k \log n/100}.
\end{aligned}$$

Thus by taking the union bound, we see that the probability of there being such sets A_1 and A_2 is at most

$$\sum_{k=1}^{n/(\log n \cdot \log \log n)} \left(\frac{e^{1000} n}{k \log n} \right)^{k \log n/500} \cdot \left(\frac{k \log n}{n} \right)^{k \log n/100} \leq \sum_{k=1}^{n/(\log n \cdot \log \log n)} \left(\frac{e^{1000} k^4 \log^4 n}{n^4} \right)^{k \log n/500}.$$

Since the summand is maximized at $k = 1$ in the range $1 \leq k \leq \frac{n}{\log n \cdot \log \log n}$, we see that the right hand side of above is $o((\log n)^{-2})$. □

We now can find a Hamilton cycle without any blue edges, and conclude the proof that **OrientPrime** succeeds *whp*.

Theorem 6.10. *There exists a Hamilton cycle with no blue edges whp.*

Proof. By Proposition 5.2, we *whp* can find a 1-factor, which by Claims 6.3 and 6.4 contains at most $\log n$ blue edges that are vertex-disjoint. By Claim 6.2, it suffices to find a Hamilton cycle after compressing the vertices in B from the 1-factor, since *whp* there are no blue edges incident to B. With slight abuse of notation, we may assume that the compressed graph contains n vertices, and that we are given at least $\frac{n \log n}{3}$ random edges over this 1-factor. By Lemma 6.5, by using half of these random edges, we can find a 1-factor consisting of cycles $C_0, C_1, \cdots, C_t$ so that $|C_0| = n - o(n)$ and $t = O(\log n)$. Suppose that there are k blue edges that belong to the 1-factor, for some $k = O(\log n)$. We still have a set of at least $\frac{n \log n}{6}$ non-revealed edges E_L that we are going to use in Lemma 6.6.

Let X be a set which we will update throughout the process. Consider the cycle C_1. If it contains a blue edge, then remove it from the cycle to obtain a path P_1. Otherwise, remove an arbitrary edge from C_1 to obtain $P_1 = (w_0, w_1, \cdots, w_a)$. Expose the set of pairs $\{(w_a, x) : x \in V(C_0), (w_a, x) \text{ is not exposed}\}$ which is of size at least $|C_0| - |X| - 2k = n - o(n)$. By Claims 6.7 and 6.8, with probability at least $1 - o((\log n)^{-2})$, the outcome is typical and there exists at least one non-blue edge of the form (w_a, x) for some $x \in V(C_0)$. Condition on this event. Note that the set of exposed pairs is a subset of $Q(\{w_a\})$, and that this gives a path P over the vertices of C_0 and P_1, which starts at w_0 and ends at some vertex in C_1 (thus w_a is not a endpoint). Add w_a to the set X, and let Y_1 be the set of vertices incident to some blue edge that belongs to C_0 or P_1. Note that X, Y_1 are disjoint, the set of exposed pairs is a subset of $Q(X)$, and neither of the two endpoints of P belong to $X \cup Y_1$. By applying Lemma 6.6 with X and $Y = Y_1$, with probability at least $1 - o((\log n)^{-1})$, we obtain a cycle that contains all the vertices of C_0 and C_1. Moreover, the pairs we further exposed will be a subset of $Q(Z_1)$ for some set Z_1 of size at most $\frac{n}{\log n \cdot \log \log n}$. Condition on this event and update X as the union of itself with Z_1. Note that by the definition of Y_1, X does not intersect any blue edge of the new cycle.

Repeat the above for cycles $C_2, C_3, \cdots, C_t$. At each step, the success probability is $1 - o((\log n)^{-1})$, and the size of X increases by at most $1 + \frac{n}{\log n \cdot \log \log n} \le \frac{2n}{\log n \cdot \log \log n}$. Since $t = O(\log n)$, we can maintain X to have size $o(n)$, and thus the process above indeed can be repeated. In the end, by the union bound, with probability $1 - o(1)$, we find a Hamiltonian cycle which has at most k blue edges. Let Y be the vertices incident to the blue edges that belong to this Hamilton cycle. Note that $|Y| \le 2k$ and $X \cap Y = \emptyset$. Remove one of the blue edges (y, z) from the cycle to obtain a Hamilton path. Apply Lemma 6.6 with the sets X and $Y \setminus \{y, z\}$ to obtain another Hamilton cycle with fewer blue edges. Since the total number of blue edges is at most $k = O(\log n)$, the blue edges are vertex-disjoint, and the probability of success is at least $1 - o((\log n)^{-1})$, after repeating this argument for all the blue edges in the original cycle, we obtain a Hamilton cycle with no blue edge. □

7 Concluding Remarks

In this paper we considered the following natural question. Consider a random edge process where at each time t a random edge (u, v) arrives. We are to give an on-line orientation to each edge at the time of its arrival. At what time t^* can one make the resulting directed graph Hamiltonian? The best that one can hope for is to have a Hamilton cycle when the last vertex of degree one disappears, and we prove that this is indeed achievable *whp*.

The main technical difficulty in the proof arose from the existence of bud vertices. These were degree-two vertices that were adjacent to a saturated vertex in the auxiliary graph $D_{5-in,5-out}$. Note that for our proof, we used the method of deferred decisions, not exposing the end-points of certain edges and leaving them as random variables. Bud vertices precluded us from doing this naively and forced us to expose the end-point of some of the edges which we wanted to keep unexposed (it is not difficult to show that without exposing these endpoints, we cannot guarantee the bud vertices to have degree at least 2). If one is willing to settle for an asymptotically tight upper bound on t^*, then one can choose $t^* = (1 + \varepsilon) n \log n / 2$, and then for $n = n(\varepsilon)$ sufficiently large there are no bud vertices.

Moreover, since for this range of t^*, the vertices will have significantly larger degree, the orienting rule can also be simplified. While not making the analysis "trivial" (i.e., an immediate consequence of the work in [14]), this will considerably simplify the proof.

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References

[1] N. Alon, J. Spencer, **The Probabilistic Method**, 2nd ed., Wiley, New York, 2000.

[2] Y. Azar, A., Broder, A. Karlin, E. Upfal, Balanced allocations, *SIAM Journal on Computing* 29 (1999), 180–200.

[3] J. Balogh, B. Bollobas, M. Krivelevich, T. Muller and M. Walters, Hamilton cycles in random geometric graphs, manuscript.

[4] T. Bohman, A. Frieze, Avoiding a giant component, *Random Structures and Algorithms* 19 (2001), 75–85.

[5] T. Bohman, A. Frieze, and N. Wormald, Avoidance of a giant component in half the edge set of a random graph, *Random Structures and Algorithms* 25 (2004), 432–449.

[6] T. Bohman and D. Kravitz, Creating a giant component, *Combinatorics, Probability, Computing* 15 (2006), 489–511.

[7] B. Bollobás, The evolution of sparse graphs, *in* "Graph theory and combinatorics proceedings, Cambridge Combinatorial Conference in Honour of Paul Erdős, 1984" (B. Bollobas, Ed.), 335–357.

[8] B. Bollobás, T. Fenner, and A. Frieze, An algorithm for finding hamilton cycles in random graphs, *in* "Proceedings, 17th Annual ACM Symposium on Theory of Computing, 1985", 430–439.

[9] C. Cooper and A. Frieze, Hamilton cycles in random graphs and directed graphs, *Random Structures and Algorithms* 16 (2000), 369–401.

[10] R. Diestel, **Graph theory**, Volume 173 of *Graduate Texts in Mathematics*, Springer-Verlag, Berlin, 3rd edition, 2005.

[11] P. Erdős and A. Renyi, On the evolution of random graphs, *Publ. Math. Inst. Hung. Acad. Sci.* 5A (1960), 17–61.

[12] P. Erdős and P. Turán, On some problems of a statistical group-theory. I, *Z. Wahrsch. Verw.* 4 (1965), 175–186.

[13] A. Flaxman, D. Gamarnik, and G. Sorkin, Embracing the giant component, *Random Structures and Algorithms* 27 (2005), 277–289.

[14] A. Frieze, An algorithm for finding hamilton cycles in random directed graphs, *Journal of Algorithms*, 9 (1988), 181–204.

[15] A. Frieze, Personal communication.

[16] J. Komlos and E. Szemerédi, Limit distribution for the existence of Hamilton cycles in random graphs, *Discrete Math*, 43 (1983), 55–63.

[17] A. Korshunov, Solution of a problem of Erdős and Rényi on Hamilton cycles non-oriented graphs, *Soviet Math. Dokl.*, 17 (1976), 760–764.

[18] M. Krivelevich, P. Loh, and B. Sudakov, Avoiding small subgraphs in Achlioptas processes, *Random Structures and Algorithms* 34 (2009), 165–195.

[19] M. Krivelevich, E. Lubetzky, and B. Sudakov, Hamiltonicity thresholds in Achlioptas processes, *Random Structures and Algorithms* 37 (2010), 1–24.

[20] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math*, 14 (1976), 359–364.

[21] R. Robinson and N. C. Wormald, Almost all regular graphs are hamiltonian, *Random Structures and Algorithms*, 5 (1994), 363–374.

[22] A. Sinclair and D. Vilenchik, Delaying Satisfiability for random 2SAT, *APPROX-RANDOM* (2010), 710–723.

[23] J. Spencer and N. Wormald, Birth control for giants, *Combinatorica 27* (2007), 587–628.