

Probabilistic method

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Assignment 3

Due: November 28 (in my mailbox or by email)

Solution of every problem should be no longer than one page!

Problem 1: The *Hajós number* of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise vertex disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number?

Problem 2: Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

Problem 3: A finite sequence $a = a_1 a_2 \dots a_n$ of symbols from a set S is called *non-repetitive* if it does not contain a sequence of the form $xx = x_1 x_2 \dots x_m x_1 x_2 \dots x_m, x_i \in S$, as a subsequence of consecutive terms. Prove that there exist *infinite* non-repetitive sequence over any 15-element set of symbols.

Problem 4: A family of subsets \mathcal{G} is called *intersecting* if $G_1 \cap G_2 \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ be k intersecting families of subsets of $\{1, 2, \dots, n\}$. Prove that

$$|\cup_{i=1}^k \mathcal{F}_i| \leq 2^n - 2^{n-k}.$$

Problem 5: Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V . Let $H = G[U]$ be the induced subgraph of G on U . Prove that

$$\text{Prob}(\chi(H) \leq 400) < 1/100.$$

Hint. Prove first that the expectation of $\chi(H)$ is at least 500.