Probabilistic method

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Assignment 3

Due: May 25 (in class, in my mailbox or by email)

Solution of every problem should be no longer than one page!

Problem 1: The *Hajós number* of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise vertex disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number ?

Problem 2: Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

Problem 3: Prove that for every $\epsilon > 0$ there is a finite $l_0 = l_0(\epsilon)$ such that for every n there is a sequence of bits $a_1, a_2, \ldots, a_n \quad a_i \in \{0, 1\}$, with the property that for every $l > l_0$ and every $i \geq 1$ the two binary vectors $u = (a_i, a_{i+1}, \ldots, a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates.

Problem 4: Show that the probabiity that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k - 1$ is at least $1/4^k$.

Problem 5: Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that

$$Prob(\chi(H) \leq 400) < 1/100.$$