Probabilistic method

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Assignment 2

Due: November 10

Solution of every problem should be no longer than one page!

Problem 1: Let X be a random variable taking integral nonnegative values, let $E(X^2)$ denote the expectation of its square, and let $Var(X)$ denote its variance. Prove that

$$Prob(X = 0) \leq \frac{Var(X)}{E(X^2)}.$$

Problem 2: Let $G = (V, E)$ be a simple graph with n vertices and m edges and let λ be the integer. Prove that

(i) There are at least $\lambda^n(1 - m/\lambda)$ proper vertex colorings of G with λ colors.

(ii) The number of proper vertex colorings of G with λ colors is at most $\lambda^n(\lambda - 1)/m$.

(iii) Show that the upper bound from part (ii) can be improved further to $\lambda^n \frac{\lambda-1}{\lambda+m-1}$

Problem 3: Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, \ldots, a_n$ satisfying $\sum_{i=1}^n a_i^2 = 1$, if $(\epsilon_1, \ldots, \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$Prob(|\sum_{i=1}^n \epsilon_i a_i| \leq 1) \geq c.$$

Problem 4: Let $v_1, v_2, \ldots, v_n$ be n vectors in R^n, each of Euclidean norm at most 1, and let $u = \sum_{i=1}^n p_i v_i$, where $0 \leq p_i \leq 1$ for all i.

(i) Prove that there are $\epsilon_i \in \{0, 1\}$ such that

$$||\sum_{i=1}^n \epsilon_i v_i - u|| \leq \sqrt{n}/2.$$

(ii) Prove that the above estimate is tight for all n.

(iii) Prove that even for $m > n$ and for $v_1, \ldots, v_m \in R^n$, each of norm at most 1, and for $u = \sum_{i=1}^m p_i v_i$ with $0 \leq p_i \leq 1$, there are $\epsilon_i \in \{0, 1\}$ such that

$$||\sum_{i=1}^m \epsilon_i v_i - u|| \leq \sqrt{n}/2.$$

Problem 5: Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \ (mod\ p) : x \in X\}$ intersects every interval of length at least p/k in $\{0, 1, \ldots, p-1\}$.
Hint. Pick random residues a and b and consider $\{ax + b \ (mod\ p) : x \in X\}$.

Problem 6: Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$.