Probabilistic method

Instructor: Benny Sudakov

Assignment 1

Due: October 21

Solution of every problem should be no longer than one page!

Problem 1: Suppose $n \geq 4$ and let H be an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ edges. Prove that there is a coloring of the vertices of H by 4 colors so that in every edge all 4 colors are represented.

Problem 2: Let $\mathcal{F}$ be a family of subsets of $N = \{1, 2, \ldots, n\}$, and suppose there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.
Hint. Consider a random permutation of the elements of N.

Problem 3: Let $G = (V, E)$ be a graph on $n \geq 10$ vertices and suppose that if we add to G any edge not in G then the number of copies of a complete graph of 10 vertices in it increases. Show that the number of edges of G is at least $8n - 36$.

Problem 4: Let $G = (V, E)$ be a bipartite graph on n vertices with a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

Problem 5: Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing sub-sequence of length at least $c\sqrt{n}$.