Combinatorics

Instructor: Benny Sudakov

Take home exam

Due: May 15, before 3.00 P.M.

Solution of every problem should be no longer than one page!
Books are not allowed on this exam, you can use only notes from the class!
Please write and sign in the end of the exam the pledge that you did not violated honor code!

Problem 1: Let $\{(A_i, B_i), 1 \le i \le h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = l$ for all i, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \le \frac{(k+l)^{k+l}}{k^k l^l}$.

Problem 2: Let G be a graph on n vertices and let $\overline{G}$ be it s complement. Let $t(G)$ denote the total number of triangles in G and $\overline{G}$. Express $t(G)$ as a function of the degrees $d_1, \ldots, d_n$ of vertices of G and prove that

$$t(G) \ge \frac{n(n-1)(n-5)}{24}.$$

Problem 3: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an n-element set with the property that $|A \cap B|$ is *odd* for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that then $|\mathcal{A}||\mathcal{B}| \le 2^{n-1}$.

Problem 4: Let A_G be the adjacency matrix of graph G which has only k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that $f(A_G) = 0$, where polynomial f equals

$$f(x) = (x - \lambda_1) \cdots (x - \lambda_k).$$

Show that the number of distinct eigenvalues of every connected graph is greater than its diameter.

Problem 5: Prove that for every positive integer r there exist $N(r)$ such that for all $n \ge N(r)$ any coloring of all subsets of $[n]$ into r colors contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same color.