Combinatorics

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Assignment 5

Due: May 2

Solution of every problem should be no longer than one page!

Problem 1: Let $\mathcal{F}$ be a collection of subsets of $[n]$ such that all members of $\mathcal{F}$ and their pairwise intersections have even size. Prove that $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$.

Problem 2: Let $A_1, \ldots, A_m$ be subsets of an n-element set. Assume that their pairwise symmetric differences $A_i \triangle A_j = (A_i \setminus A_j) \cup (A_j \setminus A_i)$ have only two sizes. Prove that $m \leq \frac{n(n+1)}{2} + 1$. Find $m = \frac{n(n-1)}{2} + 1$ subsets of an n-element set with only two sizes of symmetric differences.

Problem 3: Prove that if G is a connected graph and λ_1 is its maximum eigenvalue, then the eigenvector of eigenvalue λ_1 has strictly positive coordinates. Prove that G has only one eigenvector with eigenvalue λ_1.

Problem 4: Prove that a connected graph G with maximum eigenvalue λ_1 is bipartite if and only if $-\lambda_1$ is also an eigenvalue.

Problem 5: Let λ_1 be the maximum eigenvalue of graph G. Prove that the chromatic number of G is at most $\lambda_1 + 1$.