Combinatorics

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Assignment 4

Due: April 18

Solution of every problem should be no longer than one page!

Problem 1: A *sunflower* with k petals and core Y is a collection of sets $S_1, \ldots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$ and $S_i - Y$ are all non-empty. Let $\mathcal{F}$ be a family of sets each of cardinality s. Prove that if $|\mathcal{F}| > s!(k-1)^s$ then $\mathcal{F}$ contains a sunflower with k petals.

Problem 2: Prove that if each collection of at most $\binom{r+s}{r}$ members of an r-uniform set system can be covered by s points, then all members can.

Problem 3: Let $\mathcal{F}$ be a family of subsets of an n element set such that there are no three distinct members $F_1, F_2, F_3 \in \mathcal{F}$ which satisfy $F_1 \subset F_2 \subset F_3$. Prove that

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lceil n/2 \rceil}.$$

Problem 4: Let G_1 and G_2 be two graphs on the same vertex set V. Prove that chromatic number of the union $G_1 \cup G_2$ (we take the union of edge sets of both graphs) satisfies

$$\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2).$$

Use this to show that if $H_1, \ldots, H_t$ are bipartite graph whose union is a complete graph on n vertices, then $t \geq \log_2 n$.

Problem 5: Let $A_1, \ldots, A_m$ be subsets of an n-element set such that $|A_i|$ is not divisible by 6 for every i, but the sizes of all pairwise intersections $|A_i \cap A_j|$ are divisible by 6. Prove that $m \leq 2n$.