Combinatorics

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Assignment 3

Due: April 4

Solution of every problem should be no longer than one page!

Problem 1: Let G be a graph with average degree at least $2d$. Prove that G contains a non-empty subgraph G' with minimum degree at least d. Using this, show that if G has n vertices and kn edges than it contains every tree on k vertices as a subgraph (tree is a connected graph with no cycles).

Problem 2: Prove that every tournament T contains a Hamiltonian path, i.e., a directed path which visits every vertex of T exactly once.

Problem 3: Prove that there exists a tournament T on n vertices which contains at least $n!\,2^{-(n-1)}$ distinct Hamiltonian paths.

Problem 4: Let T be a tournament on n vertices such that for every subset of vertices U of size k there is a vertex v which dominates U, i.e., T contains all directed arcs $\{(v, u) : \; u \in U\}$.
(a) Prove that $n > 2^k$.
(b) Prove that there exists a positive constant c such that $n \geq ck2^k$.

Problem 5: Let $v_1, \ldots, v_n$ vectors in $\mathbb{R}^n$ of unit length $|v_i| = 1$. Prove that there are signs $\epsilon_i = \pm 1$ such that

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n| \leq \sqrt{n}.$$

Show that this is tight, i.e., $\sqrt{n}$ estimate can not be improved.

Problem 6: Given a hypergraph H, the transversal number of $\tau(H)$ is the minimal cardinality of a set of vertices which intersects all the edges of H. Prove that if H has n vertices and m edges all of size r then for any $p \in [0, 1]$,

$$\tau(H) \leq pn + (1-p)^r m.$$

Deduce from this that

$$\tau(H) \leq \frac{m + n \log r}{r},$$

where log is to the base $e = 2.71...$