Combinatorics

Instructor: Benny Sudakov

Assignment 2

Due: March 14

Solution of every problem should be no longer than one page!

Problem 1: Let G be a graph with n vertices and m edges. Then the number of triangles in G is at least

$$\frac{4m}{3n}\left(m-\frac{n^2}{4}\right).$$

Show that this estimate is tight (best possible) when $m = n^2/3$.

Problem 2: Let $r \geq 3$ and G be a graph on $n \geq r+1$ vertices with at least

$$\left(1-\frac{1}{r-1}\right)\frac{n^2}{2}+1$$

edges. Prove that G contains $K_{r+1}-e$, i.e., a copy of clique of size $r+1$ with one missing edge.

Problem 3: Let X be a set of n points in the plane. Prove that the number of pairs $x_i, x_j \in X$ such that the distance between x_i, x_j equals one is at most $cn^{3/2}$, for some absolute constant c.

Problem 4: Let D be a directed graph on n vertices such that the outdegree of every vertex is larger than $\log_2 n$. Prove that D contains an even directed cycle.
Hint Prove that the vertices of D can be partitioned into two parts such that every vertex from one part has out-neighbor in the other part.

Problem 5: Let $\mathcal{F}$ be a collection of subsets of X such that every two members of $\mathcal{F}$ intersect in at least two points. Prove that the vertices of X can be 2-colored so that no set in $\mathcal{F}$ is monochromatic.

Problem 6: Let $k \geq 4$ and let $\mathcal{F}$ be a collection of k-element subsets of X. Prove that if $\mathcal{F}$ has less than $\frac{4^{k-1}}{3^k}$ sets than the vertices of X can be 4-colored so that every set in $\mathcal{F}$ is *rainbow*, i.e., contains vertices of all 4 colors.