Combinatorics

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Assignment 1

Due: February 28

Solution of every problem should be no longer than one page!

Problem 1: If G is a graph with even degrees, then edges of G can be oriented in such a way that every vertex in the resulting orientation has the same indegree as outdegree.

Problem 2: Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Then there is always a set of consecutive numbers $a_k, a_{k+1}, \ldots, a_\ell$ whose sum $\sum_{i=k}^{\ell} a_i$ is divisible by n.

Problem 3: Prove that for every $k \geq 2$ there exist $n_0 = n_0(k)$ such that every coloring of $1, 2, \ldots, n_0$ in k colors contains three distinct numbers $1 \leq a, b, c \leq n_0$ which have the same color and satisfy $a \cdot b = c$.

Problem 4: A *transitive tournament* is an orientation of a complete graph for which the vertices can be numbered in such a way that (i, j) is a directed edge if and only if $i < j$.

- Show that every orientation of the complete graph K_n contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices
- Show that if $k \geq 2\log_2 n + 2$, then there is an orientation of K_n with no transitive tournament on k vertices

Problem 5: Let $g_1(x), \ldots, g_k(x)$ be bounded real functions and let $f(x)$ be another real function. Suppose that there are positive constants ϵ and δ such if $f(x) - f(y) > \epsilon$ then $\max_i(g_i(x) - g_i(y)) > \delta$. Prove that f is also bounded.

Problem 6: Prove that every set of $2^m + 1$ vectors in $\mathbb{R}^m$ with integers coordinates contains a pair of points whose mean vector has also integers coordinates.