

Algebraic Methods in Combinatorics

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Assignment 3

Due: Any time till March 16

Solution of every problem should be no longer than one page!

Problem 1: Let A and B be two non-empty subsets of \mathbb{Z}_p and let

$$X = \{a + b \mid a \in A, b \in B, ab \neq 1\}.$$

Show that $|X| \geq \min\{|A| + |B| - 3, p\}$. Show that this is sharp for $|A|, |B| \geq 2$.

Problem 2: (i) Let λ_1 be the maximum eigenvalue of graph G . Prove that the chromatic number of G is at most $\lambda_1 + 1$.

(ii) Let A_G be the adjacency matrix of graph G which has only k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Prove that $f(A_G) = 0$, where polynomial f equals

$$f(x) = (x - \lambda_1) \cdots (x - \lambda_k).$$

Show that the number of distinct eigenvalues of every connected graph is greater than its diameter.

Problem 3: (i) Let p be prime and let $n \geq d(p - 1) + 1$ for some positive integer d . Prove that if $h(x_1, \dots, x_n)$ is a polynomial with integer coefficients and degree at most d such that $h(0, \dots, 0) = 0$, then there is a nonzero vector $e \in \{0, 1\}^n$ such that $h(e) = 0 \pmod{p}$.

(ii) Prove that any hypergraph with at least $d(p - 1) + 1$ edges and maximal degree at most d contains subset of edges whose union has size $0 \pmod{p}$. Show that $d(p - 1)$ edges are not enough.

Hint. Use Inclusion-Exclusion principle.

Problem 4: (i) Let X_1 and X_2 be two disjoint sets and let r_1, r_2 and s_1, s_2 be positive integers. For $i = 1, 2$ and $1 \leq j \leq m$ let A_{ij} and B_{ij} be subset of X_i such that $|A_{ij}| \leq r_i$, $|B_{ij}| \leq s_i$ and $(A_{1j} \cup A_{2j}) \cap (B_{1j} \cup B_{2j}) = \emptyset$ but $(A_{1j} \cup A_{2j}) \cap (B_{1k} \cup B_{2k}) \neq \emptyset$ for all $1 \leq j < k \leq m$. Prove that

$$m \leq \binom{r_1 + s_1}{r_1} \binom{r_2 + s_2}{r_2}.$$

(ii) Let G be a bipartite graph with parts V_1 and V_2 of sizes a and b respectively and let $G = G_0 \subset G_1 \subset \dots \subset G_t = K_{a,b}$ be a sequence of graphs such that G_i is obtained from G_{i-1} by adding an edge joining vertices belonging to different parts. In addition, suppose that for all i , G_i contains more copies of the complete bipartite graph $K_{r,s}$ with r vertices in V_1 and s vertices in V_2 than G_{i-1} . Prove that G has at least $ab - (a - r + 1)(b - s + 1)$ edges.

Problem 5: Let p be an odd prime and let A and B be two subsets of \mathbb{Z}_p each of size k . Prove that there is an ordering $\{a_1, \dots, a_k\}$ of the elements of A and an ordering $\{b_1, \dots, b_k\}$ of the elements of B such that all the sums $a_i + b_i$ are pairwise distinct.