

# Algebraic Methods in Combinatorics

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## Assignment 3

Due: Any time till March 7

**Solution of every problem should be no longer than one page!**

**Problem 1:** Let  $A$  and  $B$  be two non-empty subsets of  $\mathbb{Z}_p$  and let

$$X = \{a + b \mid a \in A, b \in B, ab \neq 1\}.$$

Show that  $|X| \geq \min\{|A| + |B| - 3, p\}$ . Show that this is sharp for  $|A|, |B| \geq 2$ .

**Problem 2:** (i) Prove that for any connected graph with largest eigenvalue  $\lambda_1$ , the eigenvector of  $\lambda_1$  has only positive entries.

(ii) Let  $G$  be a  $d$ -regular connected graph on  $n$  vertices with diameter  $D$  and eigenvalues  $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Prove that

$$d - \lambda_2 > \frac{1}{Dn}.$$

**Problem 3:** (i) Let  $p$  be prime and let  $n \geq d(p-1) + 1$  for some positive integer  $d$ . Prove that if  $h(x_1, \dots, x_n)$  is a polynomial with integer coefficients and degree at most  $d$  such that  $h(0, \dots, 0) = 0$ , then there is a nonzero vector  $e \in \{0, 1\}^n$  such that  $h(e) = 0 \pmod{p}$ .

(ii) Prove that any hypergraph with at least  $d(p-1) + 1$  edges and maximal degree at most  $d$  contains subset of edges whose union has size  $0 \pmod{p}$ . Show that  $d(p-1)$  edges are not enough.

**Hint.** Use Inclusion-Exclusion principle.

**Problem 4:** (i) Let  $X_1$  and  $X_2$  be two disjoint sets and let  $r_1, r_2$  and  $s_1, s_2$  be positive integers. For  $i = 1, 2$  and  $1 \leq j \leq m$  let  $A_{ij}$  and  $B_{ij}$  be subset of  $X_i$  such that  $|A_{ij}| \leq r_i$ ,  $|B_{ij}| \leq s_i$  and  $(A_{1j} \cup A_{2j}) \cap (B_{1j} \cup B_{2j}) = \emptyset$  but  $(A_{1j} \cup A_{2j}) \cap (B_{1k} \cup B_{2k}) \neq \emptyset$  for all  $1 \leq j < k \leq m$ . Prove that

$$m \leq \binom{r_1 + s_1}{r_1} \binom{r_2 + s_2}{r_2}.$$

(ii) Let  $G$  be a bipartite graph with parts  $V_1$  and  $V_2$  of sizes  $a$  and  $b$  respectively and let  $G = G_0 \subset G_1 \subset \dots \subset G_t = K_{a,b}$  be a sequence of graphs such that  $G_i$  is obtained from  $G_{i-1}$  by adding an edge joining vertices belonging to different parts. In addition, suppose that for all  $i$ ,  $G_i$  contains more copies of the complete bipartite graph  $K_{r,s}$  with  $r$  vertices in  $V_1$  and  $s$  vertices in  $V_2$  than  $G_{i-1}$ . Prove that  $G$  has at least  $ab - (a-r+1)(b-s+1)$  edges.

**Problem 5:** Let  $p$  be an odd prime and let  $A$  and  $B$  be two subsets of  $\mathbb{Z}_p$  each of size  $k$ . Prove that there is an ordering  $\{a_1, \dots, a_k\}$  of the elements of  $A$  and an ordering  $\{b_1, \dots, b_k\}$  of the elements of  $B$  such that all the sums  $a_i + b_i$  are pairwise distinct.