Algebraic Methods in Combinatorics

Instructor: Benny Sudakov

Assignment 2

Due: February 23

Solution of every problem should be no longer than one page!

Problem 1: Let p be a prime and let L be a subset of $\{0, 1, \dots, p-1\}$ of size s. Suppose that $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are two families of subsets of $[n]$ satisfying
(i) $|A_i \cap B_i| \notin L(mod\ p)$ for all $1 \le i \le m$;
(ii) $|A_j \cap B_i| \in L(mod\ p)$ for all $1 \le j < i \le m$.
Then $m \le \sum_{i \le s} \binom{n}{i}$.

Problem 2: Let p be a prime, $k \ge 2$ be an integer and let L be a subset of $\{0, 1, \dots, p-1\}$ of size s. If $\mathcal{F}$ is a family of subsets of $[n]$ with the property that $|F| \notin L(mod\ p)$ for every $F \in \mathcal{F}$, but $|F_1 \cap F_2 \cap \cdots \cap F_k| \in L(mod\ p)$ for any collection of k distinct sets in $\mathcal{F}$, then

$$|\mathcal{F}| \le (k-1) \sum_{i=0}^{s} \binom{n}{i}.$$

Show that for fixed p and k this result is asymptotically tight.

Problem 3: Let A be a subset of $\mathbb{Z}_3^n$ with the property that for any two distinct vectors $a, b \in A$ there is some coordinate i such that $b_i = a_i + 1$, where addition is defined modulo 3. Prove that $|A| \le 2^n$. Find such subset A of size at least $2^{n-o(n)}$.

Problem 4: We say that a family $\mathcal{F}$ of k-element subsets of $[n]$ is a k-forest if for every $F \in \mathcal{F}$ there is a partition $[n] = V_{1,F} \cup \cdots \cup V_{k,F}$ such that F is the only member of $\mathcal{F}$ which intersects every $V_{i,F}$. Prove that $|\mathcal{F}| \le \binom{n-1}{k-1}$
Hint. Assume w.l.o.g. that $1 \in V_{1,F}$ for all F and consider the following polynomials over $\mathbb{Z}_2$

$$p_F(x) = \prod_{i=2}^{k} \sum_{j \in V_{i,F}} x_j.$$

Problem 5: Let X be an n-point set in $\mathbb{R}^d$. An X-simplex is a convex hull of some $d+1$ points of X. Prove that there exist a point $a \in \mathbb{R}^d$ contained in at least $c_d \binom{n}{d+1}$ X-simplexes, where $c_d > 0$ is a constant depending only on dimension d.
Hint. Use Tverberg's theorem.