

Algebraic Methods in Combinatorics

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Assignment 1

Due: February 9

Solution of every problem should be no longer than one page!

Problem 1: Let A_1, \dots, A_m be subsets of an n -element set and let s be a positive integer. Suppose that $|A_i|$ is not divisible by s for every i , but the sizes of all pairwise intersections $|A_i \cap A_j|$ are divisible by s . Prove that if $s = 6$ then $m \leq 2n$. More generally, prove that for every integer s there exist a constant $c(s)$ such that the above condition on sets implies that $m \leq c(s)n$.

Problem 2: Let \mathcal{A} and \mathcal{B} be families of subsets of an n -element set with the property that $|A \cap B|$ is *odd* for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that then $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

Problem 3: Let A_1, \dots, A_m be subsets of an n -element set. Assume that their pairwise symmetric differences $A_i \Delta A_j = (A_i \setminus A_j) \cup (A_j \setminus A_i)$ have only two sizes. Prove that $m \leq \frac{n(n+1)}{2} + 1$. Find $m = \frac{n(n-1)}{2} + 1$ subsets of an n -element set with only two sizes of symmetric differences.

Problem 4: n people met in the party. They noticed that for every collection of two of them the number of people in the party whom they both know is *odd*. Prove that n is an odd number.

Hint: Show that each person knows even number of people.

Problem 5: Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a family of subsets of an n -element set such every element of the ground set is contained in at least one A_i . Suppose that \mathcal{A} is not 2-colorable (i.e., every 2-coloring of the ground set makes one of the sets A_i to be monochromatic), but $\mathcal{A} \setminus A_i$ is 2-colorable for all i . Prove that $m \geq n$.