The Probabilistic Method
Homework 1

Brett Hemenway

February 7, 2008

Problem 1 *Suppose $n \geq 4$ and let H be an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ edges. Prove that there is a coloring of the vertices of H by 4 colors so that in every edge all 4 colors are represented.*

Solution Suppose we color each vertex randomly, i.e. with probability $1/4$ it gets one of each of four colors. Let us calculate the probability that for a given edge it does not receive all four colors. There are $4 \cdot 3^n$ distinct colorings using only three colors and 4^n distinct colorings, thus the probability that an edge does *not* receive the four distinct colors is $\frac{4 \cdot 3^n}{4^n}$. If H has k edges, then the union bound gives that the probability that at least one edge does not receive all four colors is less than or equal to $k \cdot \frac{4 \cdot 3^n}{4^n}$, if $k < \frac{4^{n-1}}{3^n}$, this probability is strictly less than one, so in particular there exists a coloring in which every edge receives all four colors.

Problem 2 *Let $\mathcal{F}$ be a family of subsets of $N = \{1, 2, \ldots, n\}$ and suppose there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

Hint. Consider a random permutation of the elements of N.

Solution This solution is not probabilistic, but it is short, so I hope it is not too big a violation of your principles.
Consider all chains of strictly increasing subsets of $\{1, \ldots, n\}$ of length n. If $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n$ is such a chain then $|A_1| = 1$, $|A_2| = 2$, $\ldots |A_n| = n$. Since there are n choices for A_1, $n-1$ choices for A_2 etc, there are $n!$ such chains. By the definition of $\mathcal{F}$, each element of $\mathcal{F}$ is in exactly one such chain. Now, if A is a subset of size k, it is in $k!(n-k)!$ chains, since there are $k!$ choices for the subsets contained in A, and $(n-k)!$ choices for the subsets containing A. If we let a_k denote the number of subsets of $\mathcal{F}$ of size k. Then we have the

inequality

$$\sum_{k=1}^{n} a_k k!(n-k)! \leq n!.$$

Since the number on the left is the total number of chains containing elements of $\mathcal{F}$, and the number on the right is the total number of chains. Dividing both sides by $n!$, we have

$$\sum_{k=1}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

Since $\binom{n}{k}$ is maximized when $k = \lfloor n/2 \rfloor$, we have

$$\sum_{k=1}^{n} \frac{a_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1.$$

Thus

$$|\mathcal{F}| = \sum_{k=1}^{n} a_k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Note that by letting $\mathcal{F}$ be the collection of all subsets of $\{1, \ldots, n\}$ of size $\lfloor n/2 \rfloor$ we actually achieve this bound.

Problem 3 *Let $G = (V, E)$ be a graph on $n \geq 10$ vertices and suppose that if we add to G any edge not in G then the number of copies of a complete graph of 10 vertices in it increases. Show that the number of edges of G is at least $8n - 36$.*

Solution For convenience, call the property above property A. First notice that for any graph with property A every vertex v must have at least 8 neighbors since adding an edge between v and any other vertex results in a new complete graph on 10 vertices, thus v must now be connected to at least 9 other vertices. Note that this immediately gives the lower bound $|E| \geq 4n$.

Now, we proceed by induction. Suppose G is a graph with property A, and $n = 10$, since $\binom{10}{2} = 45$, and after we add an edge to G we have at least one complete graph on 10 vertices, G must have begun with at least 44 edges, but $44 = 8 * 10 - 36$. Assume any graph with property A and $|V| \leq n$ must have $|E| \geq 8n - 36$. Now let G be a graph with property A and $|V| = n + 1$. If G has a vertex v with only 8 neighbors, then removing this vertex and all edges connected to it, results in a graph G', which still has property A since v cannot be part of any complete graph of order 10 since it had only 8 neighbors. Now, by our induction assumption, since G' must have at least $8n - 36$ edges, since G has 8 more edges than G' we are done. Thus we may assume that every vertex of G has at least 9 neigbors. Now, ...

Problem 4 *Let* $G = (V, E)$ *be a bipartite graph on* n *vertices with a list* $S(v)$ *of at least* $\log_2 n$ *colors associated with each vertex* $v \in V$. *Prove that there is a proper coloring of* G *assigning to each vertex of* v *a color from its list* $S(v)$.

Solution Since G is bipartite, its vertices can be split into two sets A and B such that all edges go from a vertex in A to a vertex in B. Thus if we can two disjoint sets of colors S_A and S_B, and color the vertices from set A with the colors S_A and the vertices in B with S_B, this will be a proper coloring. Let $S = \bigcup_{v \in V} S(v)$ be the complete list of all colors. Now, consider creating the sets S_A and S_B at random by choosing a color in S and with probability $1/2$ putting it in S_A and probability $1/2$ putting it in S_B. Now, consider a vertex in $v \in A$. The probability that there is a no color on its list in S_A is $\frac{1}{2^{|S(v)|}}$ since each color in $S(v)$ is in S_A with probability $1/2$ and they are all independent. Since $|S(V)| > \log_2 n$, the probability that v cannot be colored by some color in S_A less than $\frac{1}{n}$. Clearly the same argument holds for vertices in B. Since there are n vertices and the probability that any one vertex cannot be colored by an element in its list is less than $1/n$, the union bound gives that the probability that at least one of the vertices cannot be colored by an element in its list is strictly less than one. Thus we conclude that there exists a proper coloring where each vertex is colored by an color from its list.

Problem 5 *Prove that there is an absolute constant* $c > 0$ *with the following property. Let* A *be an* $n \times n$ *matrix with pairwise distinct entries. Then there is a permutation of the rows of* A *so that no column in the permuted matrix contains an increasing subsequence of length at least* $c\sqrt{n}$.