Appendix to the preceding paper: A rank 3 generalization of the Conjecture of Shimura and Taniyama

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October 31, 2005

The Conjecture of Shimura and Taniyama is a special case of a general philosophy according to which a motive of a certain type should correspond to a special type of automorphic forms on a reductive group. Now familiar extensions of the conjecture include (i) essentially the same statement for elliptic curves over totally real fields and (ii) the statement that for each irreducible abelian surface over $\mathbf{Q}$ there is a genus 2 Siegel modular holomorphic cusp form of weight 2 with the same degree 4 L-function. Our purpose here is to give a new *rank 3 variant* of the Shimura-Taniyama conjecture in which elliptic curves are replaced by a naive analogue, the Picard curves. These curves have recently been much studied over finite fields, but their arithmetic study in characteristic zero seems not yet to have attracted much interest. This paper is an Appendix to [Ti] which treats at length the case (ii) above.

Motives. Let K be a number field and let M be an irreducible motive defined over K with coefficients in $\overline{\mathbf{Q}}$. We recall that for l a rational prime, this means that the l-adic realization M_l of M is a $\overline{\mathbf{Q}_l}$-direct summand of a Tate-twist $H^j_l(X, \overline{\mathbf{Q}_l})(k)$ of the l-adic cohomology $H^j_l(X, \overline{\mathbf{Q}_l})$ of a smooth projective variety X defined over K. Now, if M has rank N, then it standard to conjecture that there exists a cuspidal automorphic representation π of $GL(N, \mathbf{A}_K)$ such that, at each place v, the local parameters, which are N-dimensional representations of Weil(-Deligne) groups, associated to π and M coincide. This is often stated in the weaker form: π and M have the same L-function. Conversely, one hopes that all π, such that π_∞ is *algebraic*, so arise. While fundamental, this conjecture alone does not seem to help with all problems, e.g. those concerning Selmer groups, values of L-functions, etc. Except for $N = 2$ and K totally real, there is little useful arithmetic geometry associated directly to such a π (or even its possible correspondent on an inner form of $GL(N, K)$) to help with such questions.

[1]Partially supported by NSF Grant NSF DMS-0456252

Thus, it is natural to refine the problem and ask, as have many others: for which M (and K) should there exist a Shimura variety Sh defined over K such that M (resp. M_l) is a direct summand, or even sub-quotient, of the cohomology motive $H^*(Sh, \overline{\mathbf{Q}})$ of Sh (resp. of the l-adic cohomology $H^*(Sh, \overline{\mathbf{Q}_l})$ of Sh). In general, the response to this question is quite subtle. In fact, it cannot be given without a detailed knowledge of cohomological A-packets, endoscopy, the representation r employed to compute the Hasse-Weil zeta function, and a host of local considerations, and to date all these elements have been employed in examples. Even in the case where M is a rank 2 submotive of $H^1(A)$ for A an abelian variety over a totally real field F is complex: while the L-functions of such motives should coincide with the L-functions of holomorphic Hilbert modular cusp forms of weight 2, it is known that some such motives do not themselves occur in H^1 of any Shimura variety. (The question of whether all such M occur non-effectively as Tate-twists of factors of higher cohomology of Shimura varieties remains open ([BR2]).)

This condition that M actually occur in the cohomology of the some Sh is unnecessarily strong for some interesting problems. Indeed, as in the cases of forms (elliptic and Hilbert)-modular of weight one, and of this paper, it can be useful to find an automorphic representation π, with holomorphic π_∞, such that, for a given representation R of the L-group of group G defining Sh,

1. $L_v(\pi, R; s) = L_v(M, s)$, at least at almost all finite places v
2. $L_v(\pi^{ds}, R; s) = Lv(\rho_l(\pi^{ds}), s)$ for "sufficiently many" automorphic representations π^{ds} which are discrete series at infinity, where $\rho_l(pi^{ds})$ is an l-adic representation occurring as a subquotient of the cohomology of some Shimura variety.

A full review of the current state of affairs is out of question here. Instead, we describe an open case of this problem for submotives of $H^1(A)$ of rank 3. It will suffice here to let K be a quadratic imaginary number field. Everything we do here can be extended to the more general case where K is a CM field, but a general formulation would only obscure the ideas.

Let T be a CM field containing K, and let A be an abelian variety, defined over K, endowed with an action of an order in T, all of whose elements act K-rationally on A, and such that $1 \in T$ acts on A as the identity map. Then the topological cohomology $H^1_B(A, \mathbf{Q})$ of $A(\mathbf{C})$ is a T vector space; suppose that

$$dim_T(H^1_B(A, \mathbf{Q})) = 3.$$

The tangent space $Tan(A)$ of A is a K vector space of dimension $3g$ where $[T : K] = g$. Then $Tan(A)$ is naturally a $T \otimes_{\mathbf{Q}} K$-module. Let $J_K = \{1, \rho\}$ be the set of complex embeddings of K. Let J_T be the set of complex embeddings of T. Then $J_T = J^1_T \cup J^\rho_T$ where J^1_T is all the embeddings that restrict to 1 on K and J^ρ_T is all the embeddings that restrict to ρ on K. We now assume the representation Φ of T on $Tan(A)$ is equivalent to the following:

$$\Phi \cong 2 \sum_{\sigma \in J_T^1} \sigma \oplus \sum_{\sigma \in J_T^\rho} \sigma.$$

Now let $\tau \in J_T^1$. For example, we could take $\tau = 1_T$. Then $H_B^1(A, \overline{\mathbf{Q}})$ is a 3-dimensional $T \otimes \overline{\mathbf{Q}}$-vector space, and it is canonically decomposed as a direct sum:

$$H_B^1(A, \overline{\mathbf{Q}}) = \oplus_{\sigma \in J_T} H_B^1(A)_\sigma,$$

where $H_B^1(A)_\sigma$ is the $\overline{\mathbf{Q}}$-subspace of $H_B^1(A, \overline{\mathbf{Q}})$ on which T acts via the embedding σ. Then $H_B^1(A)_\sigma$ is the realization in Betti (singular) cohomology of a motive M_σ which is defined over K and has coefficients in $\overline{\mathbf{Q}}$.

Note that, in general for irreducible motives with coefficients in a CM field T, the T-linear dual $M^\vee = Hom_T(M, \mathbf{Q})$ is T-linearly isomorphic to the motive $M^{[\rho]}(w)$. Here the operation $[\rho]$ leaves the motive itself unchanged but alters the coefficient structure by precomposing with the automorphism $\rho : T \to T$ and the (w) denotes the w-fold motivic Tate twist, where w is the weight of M.

Examples of such motives arise as follows from the *Picard curves*, currently much-studied in cryptography. Let

$$y^3 = X^4 + aX^3 + bX^2 + cX + d,$$

where a, b, c, d are in $\mathbf{Q}$, be the affine equation of a non-singular projective curve C. Then C is called a Picard curve, has genus 3, and, after base change to $K = \mathbf{Q}[\zeta_3]$, the group of 6th roots of unity acts on C via automorphisms. Hence the ring $\mathbf{Z}[\zeta_3]$ acts on the Jacobian $Jac(C)$. Thus the motive $H^1(C) \cong H^1(Jac(C))$ is defined over K and has coefficients in $T = K$. The associated decomposition of $H^1(C) \otimes \overline{\mathbf{Q}}$ into the direct sum of $H^1(C)_1$ and $H^1(C)_\rho$ is already defined over K. A calculation ([Ho]) shows that T acts on $Tan(Jac(C)) = H^0(C, \Omega^1)$ by

$$\Phi = 1 + 1 + \rho.$$

Hence $M_1 = H^1(C)_1$ is of the class considered above. Note that the complex conjugate C^ρ of C is isomorphic to C. This means that

$$M^\rho \cong M^{[\rho]}.$$

In view of the above, this means that

$$M^\vee \cong M^\rho(w),$$

where the $^\vee$ denotes the $T(= K)$-linear dual. In particular, for a prime l, we have the relation of l-adic representations of $Gal(\overline{K}/K)$:

$$M_{1,l}^\vee \cong M_{1,l}^\rho(w).$$

Here, for any group representation $N : \Gamma_K \to GL(V)$, N^ρ is defined by $N^\rho(\tau) = N(\rho\tau\rho)$; this makes no reference to any motivic origin of N.

Automorphic Forms. Now let V be a 3 dimensional K-vector space endowed with an Hermitian form $H : V \times V \to K$. Define, as usual, the associated unitary similitude group

$$G = \{g \in GL(V, K) | H(gv, gw) = \lambda(g)H(v, w) \text{for all v, w} \in V\},$$

where $\lambda(g) \in \mathbf{Q}$ is independent of v and w. Then G is a reductive algebraic group defined over $\mathbf{Q}$. Suppose that H is chosen so that G is quasi-split. We consider as usual the automorphic forms on the associated adelic groups $G(\mathbf{A_Q})$. The Shimura variety associated to G is a family surfaces ([Ho]); the zeta functions of these surfaces were computed in [LR]. The result is a formula which computes, for each cuspidal automorphic representation π of $G(\mathbf{A_Q})$, an L-function $L(\pi, s)$ which has (unramified) Euler factors of degree 3. (See [BR3], [BR1], and [LR] for discussion and formulae which arise in the Hasse-Weil zeta function computations. In [BR1] the formula is stated without reference to the Hecke theory of G, using the base change ([R]) of π from G^1 to $G \times K = GL(3, K) \times GL(1, K)$. This is convenient for exposition but perhaps misleading in general. In any case, a detailed review of the construction is not necessary here.)

Conjecture. Let M_l be an irreducible 3-dimensional l-adic representation of $Gal(\overline{K}/K)$ which occurs as a subquotient of the cohomology of a variety defined over K. Suppose that the weight of M_l is w, i.e. it occurs in H^w. Suppose that

$$(*) \qquad M_l^\vee \cong M_l^\rho(w).$$

Then there exists an L-packet Π of cuspidal automorphic representations π of $G(\mathbf{A_Q})$ such that, for almost all places v of K

$$L(\pi, s) = L(M_l, s).$$

Furthermore, at ∞, the L-packet Π_∞ is that defined by an extension to ${}^LG$ of the homomorphism $\mathbf{C}^* = W_\mathbf{C} \to GL(3, \mathbf{C})$ defined by the Hodge structure of M ([Ta]).

Remarks.

1. This conjectural correspondence takes a M_l with regular Hodge Numbers (or regular Hodge-Tate weights), where regular means that all such weights occur with multiplicity at most one) to a Π that is discrete series at ∞. It is not hard to see that Π is unique if it exists. The converse result is a Theorem: if Π is an L-packet on $G(\mathbf{A_Q})$ which is discrete series at infinity, then there exists such an M_l; in fact, M_l occurs in the cohomology of a fiber system of abelian varieties over the base Sh, and a multiple of it is the l-adic realization of a submotive there. This follows easily from the method of [BR2]. Such motives are not considered in this paper.

2. The rank 3 motives constructed above have the property that M_l has two Hodge-Tate weights the same, but not all 3 the same. In all such cases, regardless of the particular weights, the conjectural correspondence attaches an L-packet Π consisting of π that are non-degenerate limits of discrete series at infinity. There are two such representations in each such archimedian packet and one is holomorphic. Further, Π is stable, since M_l is irreducible, and hence there exists a $\pi \in \Pi$ which is holomorphic at infinity. The structure of these archimedian L-packets is briefly reviewed in [R], 12.3. In the notation of that book, the archimedian L-parameter of Π is determined by a triple (a, b, c) of integers, in which $a \geq b \geq c$, and the case in question is that where either $a = b$, or $b = c$ but not both. These integers are related to the Hodge-Tate weights of M_l by a simple rule; in particular for the case $a = 0$, $b = c = -p$ with $p > 0$, then these numbers are p, p and 0.

Using the notations of [R], we can now link the previous discussion to this one by specializing the above general conjecture.

Conjecture: Let $M = H^1(A)$ be a motive of the type discussed in the first section. Suppose that M satisfies $*$. Then there exists a cuspidal automorphic representation π of G such that

$$\pi_\infty = J_\phi^+,$$

where $\phi = \phi(a, b, c)$ with $a = 0$, $b = -1$ and $c = -1$ such that, at almost all places v of K,

$$L_v(M, s) = L_v(\pi, s).$$

.

Remarks.

1. Here J_ϕ^+ is the "lowest homomorphic limit discrete series" representation of $G(\mathbf{R})$. In classical language, the automorphic forms in question are of weight 1 relative to the scalar factor of automorphy denoted by μ in [Sh2].

2. There is a point of normalization here. We could also state the conjecture with $\phi = \phi(a, b, c)$ with $a = b = 1$ and $c = 0$. In fact, the Shimura variety defined by G does not depend on G alone. One must choose additionally a complex structure for the associated symmetric space. This is not needed (and so is not done) in [R], but a standard choice, using the convention that K is given with a complex embedding, gives the conjecture above.

3. As for classical forms of weight 1, the holomorphic vectors in these resp-resentations do not have an interpretation in terms of the cohomology of local systems on Sh. Thus, one may expect to construct Galois representations attached to these forms only by congruences to forms of higher weight. This is treated in the dissertation of K. Tignor (in preparation).

4. It is easy to construct *endoscopic* examples of these forms if one relaxes the restriction that the motives be irreducible. For example, if E is any elliptic curve $\mathbf{Q}$, then the product E^2 has endomorphisms $M_2(\mathbf{Z})$. Given another curve $E_{CM}/\mathbf{Q}$ having (normalized) CM by K over K, the H^1 of the 3-fold $E^2 \times E_{CM}$ is the type of our conjecture. In view of the theorem of Wiles and the results of [R], the sought π exists. It is not hard to see that in this case the conjecture holds (in the sense of the Local-Global Compatibility) at all places. More generally, if one starts with the motive attached to an holomorphic elliptic modular newform of weight at least 2, one can perform this construction, obtaining however a π which is holomorphic but of a higher weight. It is not hard to see that in this case the conjecture holds (in the sense of the Local-Global Compatibility) at all places.

5. It seems additional conjectural correspondences of this type, i.e. in which motives occurring in H^1 of abelian varieties are related to holomorphic forms, are very few. We hope to complete the list in a later publication.

Finally, as a further specialization, despite the redundancy, we state the conjecture for the Picard curves alone.

Conjecture: Picard curve case. Let C be a Picard curve. Define $M_1 = H^1(C)_1$ as above. Then there exists an automorphic representation π of G such that

$$\pi_\infty = J_\phi^+,$$

where $\phi = \phi(a,b,c)$ with $a = b = 1$ and $c = 0$, such that, at almost all places v of K,

$$L_v(M,s) = L_v(\pi,s).$$

.

Remarks.

1. It it would be interesting to characterize the image of the Picard curves among all representations of the type above. Is it all forms whose Hecke eigenvalues lie in $K = \mathbf{Q}[\zeta_3]$?

2. There is a list of CM Picard curves at the end of [KW]. Is the conjecture testable on some of them?

3. It is important to examine the conjecture locally at places where the Picard curves have bad, but semistable, reduction. Of course, the conjecture should really hold at all places.

4. For proof of the conjecture using deformation theory, it is helpful to know the size of the image of the l-adic. At a prime l that splits in K, the image of $Gal(\overline{K}/K)$ lies in $GL(3,\mathbf{Q}_l) \times GL(3,\mathbf{Q}_l)$, and we can identify $M_{1,l}$ with

the representation on the first factor. In [Up] some examples are given where the image of $Gal(\overline{K}/K)$ is $GL(3, \mathbf{Z}_l)$.

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