# Hilbert modular forms and the Ramanujan conjecture

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Let F be a totally real field. In this paper we study the Ramanujan Conjecture for Hilbert modular forms and the Weight-Monodromy Conjecture for the Shimura varieties attached to quaternion algebras over $F$. As a consequence, we deduce, at all finite places of the field of definition, the full automorphic description conjectured by Langlands of the zeta functions of these varieties. Concerning the first problem, our main result is the following:

**Theorem 1** *The Ramanujan conjecture holds at all finite places for any cuspidal holomorphic automorphic representation $\pi$ of $GL(2, \mathbf{A}_F)$ having weights all congruent modulo 2 and at least 2 at each infinite place of $F$.*

See below (2.2) for a more precise statement. For background, we note that the above result has been known for any such $\pi$ at all but finitely many places, and without the congruence restriction, since 1984 ([BrLa]), as a consequence of the direct local computation of the trace of Frobenius on the intersection cohomology of a Hilbert modular variety. Additionally, the local method of [Ca] is easily seen to yield the result at *all* finite places, for the forms $\pi$ which satisfy the restrictive hypothesis that either $[F : \mathbf{Q}]$ is odd or the local component $\pi_v$ is discrete series at some finite place $v$. Hence, the novel cases in Theorem 1 are essentially those of the forms $\pi$ attached to $F$ of even degree, and which belong to the principal series at all finite $v$.

To prove Theorem 1, we here proceed globally, using the fact ([Ca], [Oh], [T1], [W]) that there exist two dimensional irreducible ([BR1], [T2]) $l$-adic representations $\rho_l^T(\pi)$ of the Galois group of $\overline{F}$ over $F$ attached to such forms $\pi$. Crucial to us is the fact that these representations satisfy the Global Langlands Correspondence, i.e. that at every finite place $v$ whose residue characteristic is different from $l$, the representations of the Weil-Deligne group defined by $\pi_v$ and $\rho_l^T(\pi)$ ([Ca],[T1], [BR1], [T2], [W]) are isomorphic. Thus we get information about $\pi_v$ from that about the local Galois representation $\rho_l^T(\pi)|D_v$ whenever we realize $\rho_l^T(\pi)$, or a closely related representation $\rho_l'(\pi)$, in some $l$-adic cohomology. Many such realizations are provided by the Shimura varieties attached to inner forms of $GL(2)/F$, and to the unitary groups $GU(2)/K$ and $GU(3)/K$ where $K$ is a totally real solvable extension of $F$. Actually, to go beyond the case of

lowest discrete series at $\infty$, in order to obtain cohomological realizations of these Galois representations $\rho_l^T(\pi)$ it is necessary to consider fiber systems of abelian varieties over these unitary Shimura varieties. However, we need no explicit treatment of them here since the result is contained in [BR1]. The fact that these are realizations of $\rho_l^T$ follows from suitable local Hasse-Weil zeta function computations at all but finitely many *good places*; it is important to note that in this paper no new such computations at bad places are done.
To actually get the results, there are several overlapping methods:

A. If one of the weights is greater than 2, or if either (a) $[F : \mathbf{Q}]$ is odd or (b) there is a finite place at which $\pi$ is discrete series, the result follows easily from a basic theorem of De Jong ([DJ]), the Local Langlands Correspondence, and the classification of unitary representations of $GL(2)$ over a local field. In all these cases there is a direct realization of $\rho_l^T(\pi)$ as a subquotient of an $l$-adic cohomology group of a variety.

B. If all the weights are 2, we proceed, using a known case of Langlands functoriality, by finding a geometric realization of a Galois representation $\rho'_l(\pi)$, made using $\rho_l^T(\pi)$, and from which we can deduce crucial constraints on the Frobenius eigenvalues of $\rho_l^T(\pi)$ at an unramified place under study. While several approaches are possible, we here use one for which the L-function of $\rho'_l(\pi)$ is, after a formal base change to a field $L$, a Rankin product L- function defined by $\pi|_L$ and a Galois twist ${}^\tau\pi|_L$. Unlike case (A) above, to conclude Ramanujan by an extension of that method we use a stronger, global Ramanujan estimate ([Sha]) for $GL(2)$ which the local analytic theory cannot provide. Although several alternative constructions of $\rho'_l(\pi)$ are possible, the present method has the merit that, further developed, it enables progress on the $p$-adic analogue of the Langlands correspondence for these forms. Nevertheless, in order not to obscure the simple formal structure of the paper, we defer $p$-adic questions to a sequel.

C. If all the weights are 2, we can give (See 4.2) prove Theorem 1 by a geometric argument (found after that of B.) using the fact that the Weight-Monodromy Conjecture is a theorem for surfaces. We give both arguments since, the method of B., although a little longer, has a chance to be applicable to other cases, such as regular algbraic forms on $GL(N)$ where $N > 2$.

In this paper, we have restricted our study to the case of forms having weights all congruent modulo 2. However, the method may extend to all holomorphic forms whose weights are all at least 2 at the infinite places. A key fact, already present in [BR1], is that a suitable twist $\pi' = \pi_K \otimes \chi$ of a CM quadratic base change $\pi_K$ of $\pi$ defines motivic forms on appropriate unitary groups $GU(2)$ and $GU(3)$. Once the Global Langlands Correspondence (See below, Section 2.3)is known for these forms, the Ramanujan Conjecture will follow by the methods of this paper. One natural approach is to generalize, in the setting of those $GU(2)$ which define curves, the results of Carayol ([Ca]), and then to extend by congruences ([T1]), to the general case.

The second main goal of the paper is to provide new examples, of arbitrary dimension, and with $N$ (See the text for definitions) of many different, often highly decomposable, types, of the Weight-Monodromy Conjecture (WMC) ([D1]).

**Theorem 2** *Let $Sh_B$ be the Shimura variety attached to a quaternion algebra $B$ over a totally real field $F$. Then WMC holds for the $l$-adic cohomology of $Sh_B$ at all finite places $v$ whose residue characteristic is different from $l$.*

**Remarks.**

1. $Sh_B$ is a projective limit of varieties $Sh_{B,W}$ , where $W$ is an open compact subgroup of the finite adele group of the reductive $\mathbf{Q}$-group $G = G_B = Res_{F/\mathbf{Q}}(B*)$ associated to the multiplicative group of $B$. Each $Sh_{B,W}$ is defined over the canonically defined number field $F'$, named by Shimura the *reflex field*; the definition is recalled below in Section 3. We say that WMC holds for $Sh_B$ if it holds for each smooth variety $Sh_{B,W}$.

2. The Shimura variety is not proper exactly when $B = M_2(F)$, in which case the connected components of the $Sh_{B,W}$ are the classical Hilbert modular varieties. In this case, the theorem is understood to refer to the $l$-adic intersection cohomologies of the Baily-Borel compactification of $Sh_{B,W}$.

3. Several authors have recently made significant progress on cases of WMC involving Shimura varieties. In [It2], instances of WMC are shown for certain Shimura varieties $Sh$ associated to unitary groups. In fact, WMC is shown at places $v$ at which $Sh$ admit $p$-adic uniformization. In [DS], the $p$-adic extension of WMC is shown for a similar class of varieties: here $v$ divides $p$. As already noted, this is a case not treated at all in this paper. Finally, in [TY] Taylor and Yoshida establish WMC, by careful study of the Rapoport-Zink spectral sequence, for all Shimura varieties associated to the unitary groups defined by division algebras over a CM field which are definite at all but one infinite place. This is the key class studied in [HT], and is a vast generalization of the Shimura curves studied in [Ca]. As a consequence, WMC is true for the $l$-adic representations attached to the class of essentially self-dual regular automorphic cusp forms on $GL(N, \mathbf{A}_F)$. This result implies Theorem 1 for $\pi$ which are discrete series at some finite place, in which case the result is due to Carayol ([Ca]).

As a corollary of the above result, we achieve easily the third main goal of the paper: the proof of Langlands' conjecture ([L1]) which describes, in automorphic terms, the Frobenius semisimplification of the action of a decomposition group $D_v$ for $v$ on the $l$-adic Galois cohomology of the quaternionic Shimura varieties. Here $v$ is any finite place of the reflex field, and $l$ is a prime different from the residue characteristic of $v$. . This result completes the zeta function computations of Langlands ([L1]), Brylinksi-Labesse ([BrLa]) and Reimann ([Re2])).

**Theorem 3** *Let $B$ be a quaternion algebra over a totally real field $F$ having $B_v \cong M_2(\mathbf{R})$ for $r > 0$ infinite places $v$ of $F$. Let $F'$ be the canonical field of definition of the r-fold $Sh_B$ attached to $B$. Let $\pi'$ be a cuspidal holomorphic representation of $G) = (B \otimes \mathbf{A}_F)^*$ such that*

1. *$\pi'_v$ has weight 2 at each split infinite place,*
2. *$\pi'_v$ is one-dimensional at each ramified infinite place,*
3. *the central character $\omega$ of $\pi'$ has the form $\omega = |\cdot|^{-1}\Psi$, with a character of finite order $\Psi$.*

*Let $l$ be a rational prime. Then for each finite place $v$ of $F'$ whose residue characteristic is different from $l$, the isomorphism class of the Frobenius semisimple parameter $(\rho^*_{W,v}, N_{W,v})$ of the Weil-Deligne group $WD_v$ of $F'_v$ defined by the restriction to a decomposition group for $v$ of the action of $Gal(\overline{\mathbf{Q}}/F')$ on*

$$H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})$$

*coincides with the class of*

$$m(\pi'_f, W) r_B(\sigma(JL(\pi')_p)|_{WD_{F'_v}}),$$

*where $m(\pi'_f, W)$ is defined by*

$$dim(H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})) = 2^r m(\pi'_f, W).$$

Here, for $p$ the place of $\mathbf{Q}$ lying under $v$,

1. $JL(\pi')_p$ is the $p$-component of the cuspidal representation of $GL(2, \mathbf{A}_F)$, obtained from $\pi'$ via the Jacquet-Langlands correspondence JL.
2. $\sigma(JL(\pi')_p)$ is the homomorphism of $WD_p$ into the L-group ${}^LG$ which is, as usual, identified with the L-group of the $\mathbf{Q}$-group $R_{F/\mathbf{Q}}(GL(2))$.
3. $r_B$ is the complex representation of dimension $2^r$ defined by Langlands.
4. Let $\mathcal{H}_{\mathcal{W}}$ be the level $W$ Hecke algebra of $G$ which consists of the convolution algebra of left and right $W$ invariant compactly supported functions on $G(\mathbf{A}_f)$. Then $\pi'_{f,W}$ is the representation of $\mathcal{H}_{\mathcal{W}}$ on the subspace $(\pi'_f)^W$ of $\pi'_f$ consisting of the vectors fixed by all of $W$.

For an exposition of (2), see [BR2], 3.5, and [Ku]. For an exposition of (3), defined by Langlands ([L1]), see [BR2] esp. 5.1, 7.2. Definitions are briefly recalled as needed in the paper. Note that we are computing the L-functions as Euler products over the primes of $F'$, not as Euler products over primes of $\mathbf{Q}$.

This result may be the first verification, for some Shimura varieties of dimension greater than one, at all places and levels, of Langlands' general conjecture.

Nevertheless, for the last two theorems, the proofs are rather formal and do not involve new direct local verifications of difficult facts. On the contrary, one key principle is that the semisimplification of the *global*, i.e. $Gal(\overline{\mathbf{Q}}/F')$, Galois action on the $l$-adic cohomology of any variety in Theorem 2 is computable in simple ways from globally *irreducible* $l$-adic representations which satisfy WMC and the Global Langlands correspondence at each place. Of course this type of fact does not hold locally: the WMC concerns, for each place $v$ of $F'$, the nature of the associated indecomposable, and in general non-irreducible, Frobenius semisimple representations of the Weil-Deligne group.

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# 1 Background

## 1.1 Weil Numbers.

Let $q$ be power of a rational prime $p$. An *integral $q$-Weil number of weight* $j \in \mathbf{Z}$ is an algebraic integer $\alpha$ having the property that, for each automorphism $\sigma$ of $\overline{\mathbf{Q}}$, we have

$$|\sigma(\alpha)| = q^{j/2},$$

with a fixed $j$ independent of $\sigma$. We omit reference to $q$ or the weight $j$ when convenient. An algebraic number of the form $\beta = \alpha q^n$, for some $n \in \mathbf{Z}$ and an integral $q$-Weil Number $\alpha$ is called a *$q$-Weil number*, or simply a Weil number, if the q is clear from context. Obvious facts about Weil numbers include: (i) the $q$-Weil numbers form a group under multiplication;(ii) if $q = q_0^f$, then $\alpha$ is a $q$-Weil number of weight $j$ if and only if $\sqrt[f]{\alpha}$ is a $q_0$-Weil number of weight $j$; (iii) all roots of unity are $q$-Weil numbers of weight 0 for all $q$.

## 1.2 $l$-adic Representations.

Let $K$ be a field and let $\Gamma_K = Gal(\overline{K}/K)$ be the group of $K$-linear automorphisms of its algebraic closure $\overline{K}$, endowed with the usual topology. For a prime $l$, let $V$ be a finite dimensional vector space over $\overline{\mathbf{Q}_l}$, and let $\rho : \Gamma_K \to GL(V)$ be a homomorphism. We say that $\rho$ is an *$l$-adic representation* if there exists a finite extension $T$ of $\mathbf{Q}_l$, a $T$ vector space $V_0$, and a continuous homomorphism $\rho_0 : \Gamma_K \to GL(V_0)$ which becomes isomorphic to $\rho$ after extension of scalars on $V_0$ from $T$ to $\overline{\mathbf{Q}}_l$. We use the notation $\rho$,$V$, and $(V, \rho)$ at will to denote such a representation. An $l$-adic representation $(V, \rho)$ is called *motivic* if there is a smooth projective variety $X$ over $K$ such that $(V, \rho)$ is isomorphic to a Tate twist of a subquotient of the $\Gamma_K$-module $H^*(\overline{X}, \overline{\mathbf{Q}_l})$ where $\overline{X}$ is the scalar extension of $X$ to the algebraic closure $\overline{K}$ of $K$. Here, for a $\Gamma_K$-module $(V, \rho)$, and $m \in \mathbf{Z}$, the Tate twisted module is the pair $(V(m), \rho(m))$ where $V(m) = V$, $\rho(m) = \rho \otimes \chi_l^m$, and $\chi_l$ is the usual $l$-adic cyclotomic character.

## 1.3 Local Weil group.

For the rest of this paper, $K$ will denote a local field of characteristic 0 and residue characteristic $p$. We denote by $q$ the number of the residue field. Of course, $q$ is a power of $p$. We let $l$ be any rational prime different from $p$. We recall some basics about the Weil group $W_K \subset \Gamma_K$ of $K$. Let $I$ be the inertia subgroup of $W_K$. Then $W_K/I$ is isomorphic to the subgroup $q^{\mathbf{Z}}$ of $\mathbf{Q}^*$; the isomorphism is that induced by the homomorphism that sends an element $w$ of $W_K$ to the power $|w|$ of $q$ to which it raises the prime-to-$p$ roots of unity in the maximal unramified extension of K. Any element $\Phi$ of $W_K$ for which $|\Phi| = q^{-1}$ is called a *Frobenius*. Let $I_w$ be the subgroup of wild inertia, i.e. the maximal pro-$p$ subgroup of $I$. Let $I_t$ denote the quotient $I/I_w$ and let $W_{K,t}$ denote $W_K/I_w$. We call these groups the *tame inertia* group and the *tame Weil* group, respectively. Then $I_t$ is non-canonically isomorphic to the product

$$\prod_{l \neq p} \mathbf{Z}_l,$$

and $W_{K,t}$ is isomorphic to the semidirect product of $\mathbf{Z}$ and $I_t$ ; the action of $W_K$ on $I_t$ is given by

$$wxw^{-1} = |w|x$$

for all $x \in I_t$ and all $w \in W_K$. Choose, once and for all, an isomorphism $t = (t_l)_{l \neq p}$ of $I_t$ with

$$\prod_{l \neq p} \mathbf{Z}_l.$$

Let $(V, \rho)$ be an $l$-adic representation of $\Gamma_K$. We extend, replacing $\Gamma_K$ by $W_K$, the definition of an $l$-adic representation to $W_K$, and thus each $l$-adic representation $\rho$ of $\Gamma_K$ gives rise, by restriction, to an $l$-adic representation of $W_K$ which we also denote by $(V, \rho)$.

## 1.4 Grothendieck's Theorem

According to a basic result of Grothendieck ([ST], Appendix), there is a subgroup $J$ of finite index in $I$ such that, for $\sigma \in J$,

$$\rho(\sigma) = exp(t_l(\sigma)N)$$

where $N \in End(V)$ is a uniquely determined nilpotent endomorphism.

If we can take $J = I$ in this theorem, $(V, \rho)$ is said to be *semistable*. It is well-known that there exists a finite extension $L$ of $K$ such that $(\rho|_L, V)$ is semistable.

## 1.5 Weil-Deligne parametrization of $l$-adic representations

Fix a choice $\Phi$ of a Frobenius in $W_K$. Define, for this $\Phi$, and any $\sigma$ in $W_K$, an automorphism

$$\rho_{WD}(\sigma) = \rho(\sigma)exp(-t_l(\Phi^{-log_q(|\sigma|)}\sigma)N)$$

of $V$. Then $\sigma \to \rho_{WD}(\sigma)$ is a continuous representation of $W_K$ whose restriction to $I$ has finite image. The triple $(V, \rho_{WD}, N)$ depends on the choice of $t_l$ and $\Phi$. Such a triple $(V', \rho'_{WD}, N')$ arises from an $l$-adic representation on $V$ of $\Gamma_K$ if and only if the relation

$$\rho'_{WD}(\sigma)N'\rho'_{WD}(\sigma)^{-1} = |\sigma|N',$$

holds for all $\sigma \in W_K$. Note that $(V, \rho)$ is semistable if and only if $\rho_{WD}$ is unramified, i.e. trivial on $I$.

## 1.6 Frobenius semisimplification.

Following Deligne ([D3], 8.5), let $\rho_{WD}(\Phi) = \rho_{WD}(\Phi)^{ss}u$ be the Jordan decomposition of $\rho_{WD}(\Phi)$ as the product of a diagonalizable matrix $\rho_{WD}(\Phi)^{ss}$ and a unipotent matrix $u$. Define, for $\sigma \in W_K$,

$$\rho_{WD}^{ss}(\sigma) = \rho_{WD}(\sigma)u^{log_q(|\sigma|)}.$$

Then $\rho_{WD}^{ss}$ is a semisimple representation of $W_K$ and, for all $\sigma$, $\rho_{WD}^{ss}(\sigma)$ is semisimple. The representation $(V, \rho_{WD}^{ss})$ is called the $\Phi$-semisimplification of $(V, \rho_{WD})$ and the triple $(V, \rho_{WD}^{ss}, N)$ is called the $\Phi$-semisimplification of $(V, \rho_{WD}, N)$.

Let now $\iota_l$ be an isomorphism of $\overline{\mathbf{Q}_l}$ with the complex numbers $\mathbf{C}$. This will be fixed in any discussion, and, to avoid a cumbersome notation, we will identify $\overline{\mathbf{Q}_l}$ with $\mathbf{C}$, suppressing explicit reference to $\iota_l$. We will use $\iota_l$ to define complex representations of the Weil-Deligne group (c.f. [D3], 8.3, [Ta], 4.1, or [Roh]), via the triples $(V, \rho_{WD}^{ss}, N)$.

## 1.7 $WD_K$.

The *Weil-Deligne* group $WD_K$ of $K$ is the semidirect product of $W_K$ with $\mathbf{C}$ defined by the relation

$$\sigma z \sigma^{-1} = |\sigma| z$$

for all $\sigma \in W_K$ and $z \in \mathbf{C}$. Using $\iota_l$ we regard $V$ as a finite dimensional complex vector space (i.e. if $z \in \mathbf{C}$ and $v \in V$, we put $zv = \iota_l^{-1}(z)v$). Then $\rho_{WD}^{ss}$ is a continuous representation of $W_K$ on $V$, and $N$ is a nilpotent endomorphism of V. The complex triple $(V, \rho_{WD}^{ss}, N)$ defines, in view of (1.5), a representation $\rho^*$ of $WD_K$ by the rule

$$\rho^*((z,\sigma)) = exp(zN)\rho_{WD}^{ss}(\sigma)$$

for all $(z,\sigma) \in WD_K$. Then $\rho^*$ satisfies

(i) the restriction to $W_K$ is semisimple, and

(ii) the restriction to $\mathbf{C} = \mathbf{G}_a(\mathbf{C})$ is algebraic.

We denote the family of all complex representations satisfying (i) and (ii) by $Rep^s(WD_K)$ and denote members by pairs $(V, \rho')$; a triple giving rise to $(V, \rho')$ by the construction above is given by

$$(V, \rho'|_{W_K}, N_{\rho'})$$

where $log$ of a unipotent matrix $M$ is the standard polynomial in $M-1$ inverting exponentiation on nilpotents and

$$N_{\rho'} = log(\rho'((1,1))).$$

Henceforth an element of $Rep^s(WD_K)$ is identified with the triple it defines. Note that a member of $Rep^s(WD_K)$ is actually a semisimple representation of $WD_K$ if and only if it factors through the quotient $W_K$. A member of $Rep^s(WD_K)$ is called *semistable* if it is trivial on $I$. We denote by $Rep^{ss}(WD_K)$ the subfamily of $Rep^s(WD_K)$ consisting of semistable representations. Of course, if $(V,\rho)$ is a semistable $l$-adic representation if and only if the associated element $(V,\rho')$ of $Rep^s(WD_K)$ belongs to $Rep^{ss}(WD_K)$.

As shown in [D3], the isomorphism class of the $(V,\rho') \in Rep^s(WD_K)$ gotten from an $l$-adic representation of $\Gamma_K$ is independent of the choices of $\Phi$ and $t_l$. The class of $(V,\rho')$ does depend on the choice of $\iota_l$, but, since any $\iota_l'$ has the form $\iota_l' = \eta\iota_l$ for an automorphism $\eta$ of $\mathbf{C}$, we see that, after such a change, $\rho'$ is just replaced by the conjugate $\eta\rho'$.

## 1.8 Structure of semistable modules.

Recall that a $WD_K$-module is *indecomposable* if it cannot be written as the direct sum two proper submodules. We have the following basic structure results ([Roh]) for the members of $Rep^{ss}(WD_K)$:

(i) Any member of $Rep^{ss}(WD_K)$ is isomorphic to a direct sum of indecomposable modules, hence of $V_{\alpha,t}$'s. As such the decomposition is unique up to

re-ordering the factors, and replacing factors by isomorphic factors.
(ii) Any indecomposable member of $Rep^{ss}(WD_K)$ is isomorphic to exactly one of the form $V_{\alpha,t} = (\mathbf{C}^{t+1}, \rho_{\alpha,t}, N_t)$, where $\alpha$ is a non-zero complex number, $t$ is a non-negative integer, and $\rho_{\alpha,t}$ is the unramified representation of $W_K$ defined by the rule:

$$\rho_{\alpha,t}(\Phi) = Diag(\alpha, q^{-1}\alpha, ..., q^{-t}\alpha),$$

where $Diag$ denote diagonal matrix, and $N = (n_{ij})$, where $n_{ij} = 0$ unless $i = j+1$, in which case $n_{ij} = 1$.

### 1.9 Structure of Frobenius semisimple modules.

We have:

(i) Any member of $Rep^s(WD_K)$ is a direct sum of indecomposable submodules. As such the decomposition is unique up to re-ordering the factors, and replacing factors by isomorphic factors.

(ii) Any indecomposable representation is isomorphic to one of the form $V_{\Lambda,t} \stackrel{def}{=} \Lambda \otimes V_{q^{t/2},t}$ where $\Lambda$ (and hence t) is a uniquely determined irreducible representation of $W_K$, and any such representation is indecomposable. Such a representation is irreducible iff $t = 0$.

(iii) if $\Lambda$ is an irreducible representation of $W_K$ and $\Phi$ is any Frobenius element in $W_K$, and $\alpha$ is an eigenvalue of $\Phi$ in $\Lambda$, then $|\alpha|$ is independent of $\alpha$.

To see the last claim, note that we can find a Galois extension $L$ of $K$ such that the restriction to $WD_L \subseteq WD_K$ of $\Lambda$ is unramified, hence a direct sum of unramified characters $\chi_k$. Since $\Lambda$ is irreducible, the $\chi_k$ are permuted transitively by the natural action of $\Gamma(L/K)$. Regarding them, via local class field theory, as characters of $L^*$, and letting $\tau$ be an element of $\Gamma(L/K)$, the action is just that sending $\chi_k$ to $\chi_k \circ \tau = \chi_k$. Hence all $\chi_k$ are the same character $\chi$. Now let $\chi_0$ be an unramified character of $W_K$ such that $\chi_0 \circ N_{L/K} = \chi$, and consider the irreducible representation $\Lambda_0 = \Lambda \otimes \chi_0^{-1}$. Then the restriction to $L$ of $\Lambda_0$ is trivial, and hence $\Lambda_0$ has finite image. In particular, $\Lambda(\Phi) = \Lambda_0(\Phi)\chi_0(\Phi)$, and so each eigenvalue $\alpha$ of $\Phi$ in $\Lambda$ is of the form $\alpha = \zeta\chi_0(\Phi)$ with a root of unity $\zeta$. This proves (iii).

Let $\Lambda$ be an irreducible representation of $WD_K$. We call the real number $w(\Lambda) = 2log_q(|\alpha|)$, where $\alpha$ is any eigenvalue of any $\Phi$, the *weight* of $\Lambda$. It is independent of the choices.

### 1.10 Pure modules.

Fix an integer $j$. An indecomposable module $V_{\Lambda,t}$ for $K$ as above is *q-pure of weight j*, or simply *pure*, if

(i) the eigenvalues of $\Phi$ in $V_{\Lambda,t}$ are $q$-Weil numbers, and

(ii) $w(\Lambda) = t + j$.

By the argument at the end of the previous subsection, changing $\Phi$ will change the eigenvalues of $\Lambda(\Phi)$ only by roots of unity, and hence both conditions are independent of the choice of $\Phi$. Also, an indecomposable $V_{\Lambda,t}$ is $q_K$-pure of weight $j$ if and only if, for each finite extension $L$ of $K$, the restriction $V_{\Lambda,t}|_L$ of $V_{\Lambda,t}$ to $WD_L \subseteq WD_K$ is $q_L$-pure of (the same) weight $j$. To see this, note since the condition is obviously stable under passage from $K$ to $L$, it is enough to show the descent statement from an $L$, as above, such that $\Lambda|L$ is unramified. In this case, if $f = f(L/K)$ is the degree of the residue field extension, then $\Phi^f$ is a Frobenius element for $W_L$, and, in the above notation, $\chi(\Phi^f) = \chi_0(\Phi)^f$. Hence $\alpha = (\chi(\Phi^f))^{1/f}\zeta$, for some $f$-th root of $\chi(\Phi^f)$. Suppose now that $\chi(\Phi^f)$ is a $q_L$-Weil number of weight j. Then, since $q_L = q_K^f$, $\alpha$ is a $q_K$-Weil number of weight j also. This shows (i) holds over $K$ if it holds over $L$. To see (ii), just note that $w(\Lambda)$ is unchanged when $q_L = q_K^f$ is replaced by $q_K$ and $|\chi(\Phi^f)|$ is replaced by $|(\chi(\Phi^f))^{1/f}|$. This proves the claim.

We say that a general member $V$ of $Rep^s(WD_K)$ is pure of weight $w$ ($w \in \mathbf{R}$) if each indecomposable constituent is pure of weight $w$. Of course, if the module $V$ is pure of weight $w$, then $w$ is uniquely determined. Furthermore, if $V$ is pure of weight $w$, then any conjugate $\eta V$, for $\eta \in Aut(\mathbf{C})$ is also pure of weight $w$.

Finally, we say that an $l$-adic representation $V$ of $\Gamma_K$ is

(i) $q$-pure of weight $w$ if one, and hence any, associated member of $Rep^s(WD_K)$ is $q$-pure of weight w, and

(ii) pure if it is $q$-pure of weight w for some w.

Here are summarized some basic facts about elements of $Rep^s(WD_K)$ :

**Proposition 1**
Let $V_1, ...., V_n$ be $l$-adic representations of $\Gamma_K$.

(i)Let $V$ be the direct sum of the $V_i$. Then $V$ is $q$-pure of weight $j$ if and only if each $V_i$ is $q$-pure of weight $j$.

(ii) $V$ is $q$-pure of weight $j$ if and only if its contragredient $V^*$ is $q$-pure of weight $-j$.

(iii) if $V$ is $q$-pure of weight $j$, then the Tate twisted module $V(m) = V \otimes \chi_l^m$, where $\chi_l$ is the usual $l$-adic cyclotomic character, and $m \in \mathbf{Z}$, is $q$-pure of weight

$j - 2m$.

(iv) If $V$ and $W$ are $q$-pure of weights $k$ and $l$, their tensor product $V \otimes W$ is $q$-pure of weight $k + l$.

### 1.11 Weight-Monodromy Conjecture.

This is the following statement([Il]):

> Let $X$ be a projective smooth variety defined over the local field $K$. Let, as usual, for a rational prime $l$ which is different from the residue characteristic of $K$, $H^j(\overline{X}, \mathbf{Q}_l)$ be the $l$-adic étale cohomology of $\overline{X}$, regarded as a $\Gamma_K$-module. Then the $\Gamma_K$-module $H^j(\overline{X}, \mathbf{Q}_l)$ is $q$-pure of weight $j$, where $q$ is the cardinality of the residue field of $K$.

Remark. Since $X$ is projective, its cohomology is polarizable, and so $\Phi$ acts on $det(H^j(\overline{X}, \mathbf{Q}_l))$ as $q^{jb_j/2}$ where $b_j$ is the dimension of $H^j(\overline{X}, \mathbf{Q}_l)$. On the other hand, if $H^j(\overline{X}, \mathbf{Q}_l)$ is $q$-pure of some weight $k$, we have also that $\Phi$ acts on this space as $q^{kb_j/2}$. Hence, we say simply that $H^j(\overline{X}, \mathbf{Q}_l)$ satisfies WMC if it is $q$-pure.

If, contrary to the convention of this paper, K and its residue field both have characteristic p, then WMC is a theorem of Deligne ([D4], Theorem 1.8.4). In mixed characteristic, WMC is known for curves and abelian varieties ([SGA7-I]), for surfaces ([RZ], Theorem 2.13, [DJ], and see below, (4.3)), certain threefolds ([It1]) and, as mentioned in the introduction, a class of Shimura varieties associated to division algebras over CM fields ([It2], [TY]).

As De Jong remarks in the Introduction to [DJ], it follows from his theory of alterations that condition (i) of the definition of purity is always satisfied for l-adic representations that are subquotients of the $l$-adic étale cohomology of a quasiprojective-variety $X$ over a non-archimedean local field $K$. We sketch this result, for the case that $X$ is smooth and projective, since it is basic.

**Proposition 2.**(De Jong). Let $X$ be a smooth projective variety defined over the local field $K$. Let $\Phi \in \Gamma_K$ be a Frobenius. Then the eigenvalues of $\Phi$ on $H^j(\overline{X}, \overline{\mathbf{Q}_l})$ are integral Weil numbers.

**Proof.** Let $L$ be a finite extension of $K$ over which there exists an $L$-alteration $a : X' \to X_L$ such that $X'$ is the generic fiber of a strictly semistable scheme $\mathcal{X}'$ defined over the ring of integers $\mathcal{O}_{\mathcal{L}}$ of $L$. Since an alteration is surjective and generically finite, we may regard $H^j(\overline{X}, \overline{\mathbf{Q}_l})$ as a submodule of $H^j(\overline{X'}, \overline{\mathbf{Q}_l})$ via $a^*$. Let $\underline{\mathcal{X}'}$ be the geometric special fiber of $\mathcal{X}'$. Since $\mathcal{X}'$ is strictly semistable, its cohomology is computable via the $\Gamma_L$-equivariant weight spectral sequence of Rapoport and Zink (c.f.[RZ], Section 2, and [Il],3.8). In the notation of [Il], we have

$$_WE_1^{ij} = H^{i+j}(\underline{\mathcal{X}'}, gr_{-i}^W R\Psi(\overline{\mathbf{Q}_l}) \Longrightarrow H^{i+j}(\overline{X'}, \overline{\mathbf{Q}_l}).$$

Thus, it suffices to show that the eigenvalues of $\Phi$ on each $_WE_1^{ij}$ are integral Weil numbers. But each $_WE_1^{i,j}$ is a direct sum of cohomology groups of the form

$$H^{j+i-2l}(\underline{\mathcal{X}'}^{(2l-i+1)}, \overline{\mathbf{Q}_l})(i-l)$$

where $l \geq max(0,i)$, and $\underline{\mathcal{X}'}^{(2l-i+1)}$ is the disjoint union of smooth proper subvarieties of $\underline{\mathcal{X}'}$ defined by taking $(2l-i+1)$- fold intersections of the irreducible divisors provided in the definition of the strict semistability of $\mathcal{X}'$. The result now follows from the Weil conjectures.

## 1.12 Weight-Monodromy: Background Facts

Let $W$ be a finite set of $q$-Weil numbers and $m : W \to \mathbf{Z}_{\geq 0}$ be a function. For us, $m(\alpha)$ is the multiplicity of $\alpha$ in the spectrum of a Frobenius in a semistable module. The pair $(W,m)$ is said to be *wm-q-pure of weight j* if ,

(i) whenever $|\alpha| > q^{j/2}$, $m(q^{-1}\alpha) \geq m(\alpha)$, and,

(ii) for all $\alpha$, $m(\alpha) = m(q^{-s_\alpha}\alpha)$, where $s_\alpha = 2log_q(|\alpha|q^{-j/2})$.

Let, for $\alpha \in W$, $|\alpha| \geq q^{j/2}$,

$$\delta(\alpha) = m(\alpha) - m(q\alpha).$$

Let $W^+$ be the subset of $W$ of all $\alpha$ such that $|\alpha| \geq q^{j/2}$.

If $(V,\rho)$ is a $q$-pure of weight j semistable representation of $WD_K$, let $b(V)$ denote the number of indecomposable factors in any representation of $V$ as a direct sum of such so that $b(V) = dim(ker(N_\rho))$. More generally, for any nilpotent endomorphism $N$ of $V$, let $b(N) = dim(ker(N))$ denote the number of indecomposable Jordan blocks in the representation of $N$ as a direct sum of such. Evidently, if $V$ is a $q$-pure of weight j semistable representation of $WD_K$, the associated pair $(W_V, m_V)$ is wm-$q$-pure of weight j. Conversely, if $(W,m)$ is wm-$q$-pure of weight j, then

$$\bigoplus_{\alpha\in W^+} V_{\alpha,s_\alpha}^{\delta(\alpha)}$$

belongs to the unique isomorphism class of $q$-pure of weight $j$ semistable representations $(V_{(W,m)}, \rho_{(W,m)})$ of $WD_K$ that give rise to $(W,m)$.
In this case let

$$b(W,m) = b(V_W) = \Sigma_{\alpha\in W^+}\delta(\alpha)$$

.
The following elementary result is key to our work in this paper.

**Proposition 3.** Let $K$ be a local field and let $V$ be a finite dimensional representation in $Rep^s(WD_K)$. Let $F^{\cdot}V$ be a filtration of $V$ by $WD_K$-stable submodules. Suppose that the graded Galois module $Gr_F(V)$ is $q$-pure of weight $j$. Then $V$ is $q$-pure of weight $j$.
**Proof.** Restricting from $K$ to a suitable extension $L$, we can assume that $V$ is semistable.

Let $E_V$ be an endomorphism of a vector space $V$, and suppose that we have an $E_V$ stable short exact sequence

$$0 \to S \to V \to Q \to 0$$

with induced endomorphisms $E_S$ and $E_Q$ on $S$ and $Q$. Let $K_S$, $K_V$, and $K_Q$ be the kernels of these operators. Then

$$dim(K_S) + dim(K_Q) \geq dim(K_V).$$

This is evident since we have a short exact sequence

$$0 \to K_V \cap S \to K_V \to \frac{K_V + S}{S} \to 0$$

and $K_V \cap S = K_S$ and $\frac{K_V+S}{S}$ is a subspace of $K_Q$.
Hence, by induction, if $E_V$ is a filtered endomorphism of $F^{\cdot}V$, inducing $Gr_F(E_V)$ on $Gr_F(V)$ , then

$$dim(ker(Gr_F(E_V))) \geq dim(ker(E_V)).$$

We apply this to the case $E_V = N$.

**Lemma**. Let $(W, m)$ be wm-$q$-pure of weight j. Let $(V, \rho)$ be a semistable representation of $W_K$ such that $(W_V, m_V) = (W, m)$. Then

$$b(V) \geq b(W, m).$$

Further,

$$b(V) = b(W, m)$$

if and only if $(V, \rho)$ is $q$-pure of weight $j$.

**Proof.** Obvious.

To conclude the proof of the Proposition, we note that $Gr_F(V)$ defines the same pair $(W, m)$ as $V$. Since we assume $Gr_F(V)$ is pure, we have $b(W, m) = b(Gr_F(V))$. By the remarks just above, we always have $b(Gr_F(V)) \geq b(V)$ and

$b(V) \geq b(W,m)$. Hence $b(V) = b(W,m)$ and so $V$ is $q$-pure of weight j.

### 1.13 A problem on abelian varieties.

**Proposition 4.**
Let $A$ be an abelian variety defined over a number field $J$. Let $M$ be an irreducible motive, defined over $J$ in the category of motives for absolute Hodge cycles generated by $A$([DM]). Then for each prime $l$ and each finite place $v$ of $J$, the $l$-adic cohomology $M_l$ of $M$ satisfies the WMC.

**Proof**. This is, of course, trivial: any irreducible $M$ is of then form $M_0(n)$ where $M_0$ is a submotive of the motive $\otimes^k H^1(A, \overline{Q})$, $k$ is a non-negative integer, and $n$ is the $n$-fold Tate twist. The $l$-adic cohomology $M_{0,l}$ of $M_0$ is a $Gal(\overline{J}/J)$ direct summand of $\otimes^k H^1(A, \overline{Q}_l)$ and hence everywhere locally satisfies WMC since $H^1(A, \overline{Q}_l)$ does.

Problem: Is the conjugacy class of $N_l$(in $GL(M_l)$) independent of $l$? Evidently, this amounts to asking whether the Frobenius eigenvalues on the semisimplification of $M_l$ is independent of $l$. Of course, these statements are consequences of the standard l-independence conjecture of Serre and Tate which asserts that, for any motive $M$ over $J$, and any non-archimedean completion $J_v = K$, the isomorphism classes of the elements of $Rep^s(WD_K)$ gotten from the $l$-adic étale cohomology groups of $M$ are all the same.

## 2 Automorphic Forms.

### 2.1 Basic Conventions

Let $F$ be a number field with adele ring $\mathbf{A}_F$. Let $A_0(F,n)$ be the set of irreducible cuspidal unitary summands of the space $L^2(GL(n, \mathbf{A}_F)/GL(n, F))$. Each constituent $\pi$ of such a space is isomorphic to a restricted tensor product $\pi = \otimes_v \pi_v$ where $\pi_v$ is an infinite dimensional irreducible unitary representation. If $v$ is finite, each $\pi_v$ is classified up to isomorphism by an associated isomorphism class $\sigma(\pi_v)$ of n- dimensional members of $Rep^s(WD_v)$, where we denote by $WD_v$ the Weil- Deligne group of $F_v$ ([HT],[Ku]). As is customary, we denote also by $\sigma(\pi_v)$ any member of its class. Let $W_v$ be the Weil group of $F_v$. Then $\sigma(\pi_v)$ is isomorphic, as in 1.9, to a direct sum of indecomposable modules of the form $V_{\Lambda_i,t} = \Lambda_i \otimes V_{q^{t/2},t}$, $1 \leq i \leq n_v$, with irreducible representations $\Lambda_i$ of $W_v$.

### 2.2 Ramanujan Conjecture.

This is the assertion:

Let $\pi \in A_0(F, n)$. Let $v$ be a finite place of $F$ and define, as above, the set of representations $\Lambda_i$ of $W_v$ for $\pi_v$. Then the image of each $\Lambda_i$ is bounded.

**Remark**: This form of the conjecture is equivalent to the more elementary statement, independent of the Local Langlands Correspondence, which asserts that each $\pi_v$ is tempered. However, we work exclusively with the formulation via Weil-Deligne groups in this paper.

Suppose now that $n = 2$. Then, at non-archimedian $v$, the local components $\pi_v$ of a cuspidal $\pi$ are classified into several types:

(i) $\pi_v$ is supercuspidal,

(ii) $\pi_v$ is a twist of the Steinberg representation: $\pi_v = St_v \otimes \psi(det)$, so that $\sigma(\pi_v) = \psi \otimes V_{q^{1/2},1}$ with a character $\psi$ of $F_v^* = W_v^{ab}$.

(iii) $\pi_v$ is principal series.

In cases (i) and (ii), $\sigma(\pi_v)$ is indecomposable and if $\pi$ is unitary, $\Lambda = \Lambda_1$ is bounded. (In case (i), $\Lambda$ is irreducible, $t = 0$, and $det(\Lambda)$ is the unitary central character of $\pi$, so $\Lambda$ is bounded; in case (ii), $t = 1$, so $\Lambda = \psi$ is one-dimensional, and $\Lambda^2$ is the unitary central character of $\pi$, so $\Lambda$ has bounded image.

For case (iii), $\sigma(\pi_v)$ is a direct sum of 2 quasicharacters $\psi_1$ and $\psi_2$ of $F_v^*$ whose product is the central character of $\pi$, hence unitary. The classification of unitary representations shows that either (a) $|\psi_1| = |\psi_2| = 1$ or (b) there are quasicharacters $\mu = |\cdot|^t$ where $0 < t < 1/2$ and $\psi$ of $F_v^*$ such that $\sigma(\pi_v)$ is the sum of quasicharacters $\mu\psi$ and $\mu^{-1}\psi$. Hence, the Ramanujan Conjecture amounts to the assertion that representations of this type (*complementary series*) don't occur as local components of cusp forms. Note that at such a place, the local central character $\omega_{\pi,v}$ of $\pi$ is $\psi^2$. For the forms of interest in this paper, $F$ is totally real, and the infinity type $\pi_\infty$ of $\pi$ is discrete series and has the property that the idele class character $\omega_\pi$ takes the form $\omega_\pi = \nu_F^j \otimes \phi$, where $\nu_F$ is the norm, $j$ is an integer, and $\phi$ is a character of finite order. Hence, $\psi^2 = \phi_v$ and so $\psi$ has finite order. Invoking the Gruenwald-Hasse-Wang theorem, we see that there is an idele class character of finite order $\eta$ such that the local identity $\eta_v = \psi$ holds. Thus, replacing $\pi$ by a form of the same infinity type $\pi' = \pi \otimes \eta^{-1}$, we see that to establish the conjecture for all local components of all cusp forms $\pi'$ of the given discrete series infinity type, it is enough to prove it for all $\pi$ of the given type at all $v$ that are unramified for $\pi$. Although this easy argument is special to $GL(2)$, it may be worth noting that solvable base change for $GL(n)$ should provide a reduction of Ramanujan to the case of semistable representations (i.e. to those whose local components $\sigma(\pi_v)$ are semistable.)

## 2.3 Global Langlands Correspondence.

Let $F$ be a number field and let $(V_l, \rho_l)$ be an irreducible n-dimensional $l$-adic representation of $\Gamma_F$. Fix, for the rest of the paper, an isomorphism $\iota_l : \overline{\mathbf{Q}_l} \to \mathbf{C}$. For each finite place $v$ of $F$, whose residue characteristic is different from $l$, choose $t_l$ and $\Phi_l$, as before. Let $\rho^*_{l,v}$ be the associated member of $Rep^s(WD_v)$ so defined.

**Global Langlands Correspondence(GLC).** This is the assertion:

> Suppose that the irreducible $l$-adic representation $(V_l, \rho_l)$ is motivic. Then there are cuspidal representations $\pi \in A_0(F, n)$ and $\chi \in A_0(F, 1)$ such that, for all $v$ whose residue characteristic is different from $l$, $\sigma(\pi_v) \otimes \chi_v$ is the class of $\rho^*_{l,v}$ .

Remark 1. Since the statement of the GLC presupposes the existence of the Local Langlands Correspondence and an $l$-adic representation of the absolute Galois group of a global field, the GLC is often called the problem of *Local-Global Compatibility.*

Remark 2. If the residue characteristic of $v$ is $l$ the classes $\sigma(\pi_v) \otimes \chi_v$ can be predicted using methods of $p$-adic cohomology ([Fo]). This done, the above conjecture is extended to all finite places.

Remark 3. It is usual, especially to treat compatibility questions as l and $\iota_l$ vary, to formulate the conjecture in terms of a motive M and its Galois representations. However, as we do not treat compatibility questions in any essential way in this paper, there is no benefit to this viewpoint.

Remark 4. There is a converse conjecture: if $\pi_\infty$ is algebraic ([C1]) then there should exist a $(V_l, \rho_l)$ corresponding to $\pi$ as above.

## 2.4 GLC and WMC

**Proposition 5**
Suppose that the GLC holds for the motivic $l$-adic representation $(V_l, \rho_l)$ over $F$. Then WMC holds for $(V_l, \rho_l)$.

**Proof.**
The conjecture is invariant under Tate twist, so we may assume that $(V_l, \rho_l)$ is isomorphic to a subquotient of $H^i(X, \overline{\mathbf{Q}}_l)$ for some smooth projective $X$ over $F$. For almost all places $v$, $\pi_v$ is unramified. At such a place, the parameter $\sigma(\pi_v)$ consists of $n = dim(V_l)$ unramified quasicharacters of $F_v^*$, whose values on a prime element of $F_v$ determine the unordered n-tuple $\{\alpha_j | j = 1, ..., n\}$. Each $\alpha_j$ is a Weil number, and, if we further restrict $v$ to be a place of good reduction of $X$, then we have $|\alpha_j| = q^{i/2}$ for all $j$, for some $i$ which is independent of $j$. Consider the cuspidal representation $\pi' = \pi \otimes |\cdot|^{i/2}$. Then $\pi'$ is unitary because

its central character is unitary; this holds at all unramified places $v$ and hence everywhere. Let $v_0$ be finite place which we wish to study. The classification ([Tad], see [Ku])of unitary representations of $GL(n, F_{v_0})$ shows that $\sigma(\pi'_{v_0})$ is a direct sum of indecomposables $\Lambda \otimes V_{q^{t/2},t}$ where

$$-1/2 < w(\Lambda) < 1/2.$$

Hence

$$(i-1)/2 < w(\Lambda \otimes |\cdot|^{-i/2}) < (i+1)/2.$$

Since $(V_l, \rho_l)$ is motivic, Proposition 1 shows that $w(\Lambda \otimes |\cdot|^{-i/2})$ is an integer in this interval. Hence $w(\Lambda \otimes |\cdot|^{-i/2}) = i$, which means $w(\Lambda) = 0$, as was to be shown.

Remark. The proof of Proposition 5 uses only the fact that the weight of $\Lambda$ is an integer, not the fact that the eigenvalues of Frobenius are algebraic numbers.

# 3 Zeta functions of quaternionic Shimura varieties.

Assume henceforth that $F$ is totally real and let $G$ be an inner form of $GL(2)/F$, so that $G(F) = B^*$ with a quaternion algebra $B$ over $F$. Let $J_{F,nc} = \{\tau_1, ..., \tau_r\}$ be the set of real embeddings (= infinite places) of $F$ where $B$ is indefinite; assume that $J_{F,nc}$ is non-empty and contains $\tau_1 = 1_F$. To $B$ is attached a Shimura variety $Sh_B$ defined over $F'$, the smallest extension of $\mathbf{Q}$ containing all elements $\tau_1(f) + ... + \tau_r(f)$ for all $f \in F$. See ([Shi], [D2]) for constructions of $Sh_B$. It is the projective limit of quasi-projective r-folds $Sh_{B,W}$, where $W$ is an open compact subgroup of $G(\mathbf{A}_{F,f}) = (B \otimes \mathbf{A}_{F,f})^*$. Each $Sh_{B,W}$ is defined over $F'$, and is a finite disjoint union of connected r- folds. These components are proper if $G$ is not $GL(2)/F$ and smooth if $W$ is small enough. Any such Shimura variety is called a *quaternionic Shimura variety.* The Hasse-Weil zeta function, at almost all places, of the $l$-adic étale cohomology of such Shimura varieties has been computed by Reimann (See [Re1], Theorem 11.6), in the case $B \neq GL(2)/F$, and Brylinski and Labesse in the case $G = GL(2)/F$. In the latter case, it is the zeta function of an intersection cohomology which has been computed, and it is this cohomology that we consider in the following, using the same notation as the other cases. The zeta functions of the $l$-adic cohomology groups of $Sh_{B,W}$ have, at almost all places of $F'$, the form conjectured by Langlands ([L1])and proved by him in the case $r = [F : \mathbf{Q}]$ ([L1]). See ([BR2], Sections 3.5, 5.1, and 7.2) for an expository treatment of the result but not the proof.

For our purposes, it is sufficient to give a global description of the result over a Galois extension $L$ of $\mathbf{Q}$ which contains $F$. Thus, for each $j \in \{1, ..., r\}$, let $\overline{\tau_j}$ be an extension of $\tau_j$ to $\overline{L}$. Let $\pi$ be a cuspidal automorphic representation of weight $(2,...,2)$ of $GL(2,\mathbf{A}_F)$ which is discrete series at any finite place of $F$ at which $B$ is ramified. Choose $\pi$ so that its central character $\omega_\pi$ satisfies $\omega_\pi = \Psi|\cdot|^{-1}$ with a character $\Psi$ of finite order. Let $T$ be the number field generated by the traces $tr(\sigma(\pi_v)(\Phi))$ for all $v$ which are unramified for $\pi$. As shown by Taylor ([T1], [T2]), there is an irreducible two-dimensional $l$-adic representation $\rho_l^T$, depending only on $\pi$ and $\iota_l$, which satisfies GLC relative to $\pi$. Let $\rho_{l,L}^T$ be the restriction to $\Gamma_L$ of $\rho_l^T$. Let ${}^{[\overline{\tau_j}]}\rho_{l,L}^T$ be the representation defined, for $\eta \in \Gamma_L$, by

$$ {}^{[\overline{\tau_j}]}\rho_{l,L}^T(\eta) = \rho_{l,L}^T(\overline{\tau_j}\eta\overline{\tau_j}^{-1}). $$

Let

$$ R_l(\pi) = R_{l,J_{F,nc}}(\pi) = \otimes_{j=1,\ldots,r}{}^{[\overline{\tau_j}]}\rho_{l,L}^T. $$

Then $R_l(\pi)$ is a semisimple $l$-adic representation of $\Gamma_L$ of dimension $2^r$.

### 3.1 Semisimple cohomology of $Sh_B$.

Let, if it exists, $\pi'$ be an automorphic representation of $G(\mathbf{A}_F)$ such that $\pi'_v$ is isomorphic to $\pi_v$ at all places $v$ of $F$ which are unramified for $B$. Thus $\pi = JL(\pi')$, where JL denotes the Jacquet-Langlands correspondence. Choose an open compact subgroup $W$ as above so that $Sh_{B,W}$ is smooth and $\pi'_f$ has a non-zero space of $W$-invariants. Let ${\pi'}_{f,W}$ denote the representation of the level $W$ Hecke algebra $\mathcal{H}_W$ gotten from $\pi'_f$. Let

$$ H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})({\pi'}_{f,W}) $$

be the ${\pi'}_{f,W}$-isotypic component of $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})$.

**Proposition 6**
The irreducible subquotients of the action of $\Gamma_L$ on $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})({\pi'}_{f,W})|L$ are exactly the irreducible subquotients of $R_l(JL(\pi'))$.

**Proof.** By the $l$-adic Cebotarev Theorem ([Se]), it suffices to show that the semisimplification of the Galois action on $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})({\pi'}_{f,W})|L$ is a multiple of $R_l(JL(\pi'))$. But, up to notation and the base change to $L$, this is given by Theorem 11.6 of [Re1], and by the main theorem of [BrLa] in the non-compact case. See Section 5.3 for some explicit review of the zeta function.

# 4 Ramanujan and Weight-Monodromy for Hilbert modular forms

Let $F$ be a totally real field and let $\pi = \pi_\infty \otimes \pi_f$ be a holomorphic cuspidal automorphic representation of $GL(2,\mathbf{A}_F)$. Up to twist, the isomorphism class of $\pi$ at the infinite places of $F$ is specified, as usual, by a tuple of positive integral weights $k = (k(\tau))$, where the variable $\tau$ runs over the real embeddings of $F$.

We normalize $\pi$ by assuming that its central character $\omega_\pi$ satisfies $|\omega_\pi| = |\cdot|^{1-k}$ where $k$ is the maximum of the $k(\tau)$'s. It is natural to classify the holomorphic cuspidal $\pi$'s into several types, depending on $\pi_\infty$, i.e. on the classical weights at each infinite place:

(i) type G: all the weights are 1.

(ii) type MC: all the weights are at least 2 and they are all congruent modulo 2;

(iii) type NMC: all the weights are at least 2 and they are not all congruent modulo 2;

(iv) type NC: at least one, but not all , of the weights are 1.

Types G and MC are well-studied. RC is known at all places for type G ([W], [RT]); there is a 2-dimensional Artin representation $\rho$ of $\Gamma_F$ that satisfies the GLC. In this paper we prove RC at all places for the class of forms MC. As mentioned in the Introduction, the method of this paper should apply to type NMC but we do not consider this case in the paper. Type NC, except for the case of CM forms of this type, is completely open. Even in the case where the weights are all congruent mod 2, we do not know any motivic realization of associated Galois representations ([J]).

### 4.1 Proof of Theorem 1.

We must show that the Ramanujan Conjecture (RC) holds for all Hilbert modular forms of type MC.

Let $\pi$ be a representation of type MC of classical weight $k = (k(\tau))$. Let $T$ be the field generated by almost all Hecke eigenvalues of $\pi$. Let $\rho_l^T$ be one of the $[T : \mathbf{Q}]$ two dimensional $l$-adic representations attached to $\pi$ which satisfy GLC. As shown in [BR1], these representations are motivic except possibly in the case where $[F : \mathbf{Q}]$ is even, $k(\tau) = 2$ for all $\tau$, and $\pi_v$ belongs to the principal series for all finite $v$. Hence, except in this case, RC follows from Proposition 5.

Let $v$ be a finite place at which we will prove that $\pi_v$ satisfies RC. Changing $l$, if necessary, we assume that $v$ does not lie above $l$. Replacing $\pi$ by a twist $\pi \otimes \Psi$, we may assume that $\pi$ is unramified at all finite places of $F$ which lie above the rational prime $p$ under $v$ . Let $\tau_1$ be the tautological infinite place of $F$ and let $\tau_2$ be another infinite place. Let $B$ be the quaternion algebra over $F$ which is unramified at $\tau_1$, $\tau_2$ , and at all finite places, and which is ramified at the remaining infinite places. Let $G$ be the inner form of $GL(2)$ over $F$ such that $G(F) = B^*$. Let $L$ be a Galois extension of $\mathbf{Q}$ which contains $F$. By Proposition 6, the 4-dimensional l- adic representation $R_l(\pi)$ of $\Gamma_L$, made using $\rho_l^T|_L$ with $J_{F,nc} = \{\tau_1, \tau_2\}$, is isomorphic to the sum of irreducible subquotients of the cohomology $H^2(Sh_{B,W}, \overline{\mathbf{Q}_l})$ of the quaterionic Shimura surface $Sh_{B,W}$, for small enough open compact subgroup $W$ of the finite adele group $(B \otimes \mathbf{A}_{F,f})^*$.

Now we need to make explicit the action on $R_l(\pi)$ of a decomposition group of a place $w$ of $L$ dividing $v$. Choose a decomposition group $D_{\overline{w}} \subset \Gamma_F$ for $w$, and denote by $R_{l,w}(\pi)$ the restriction of $\rho_l^T|_L$ to $D_{\overline{w}} \cap \Gamma_L$. Let $\tau_2 v$ be the place of $F$ lying below $\overline{\tau_2}w$. Let $f_1$ and $f_2$ be degrees of the residue field extensions associated to $L_w/F_v$, and $L_{\overline{\tau_2}w}/F_{\tau_2 v}$, respectively. For each place $v'$ of $F$ above $p$, the restriction of $\rho_l^T|L$ to $D_{v'}$ is unramified. Denote the eigenvalues of $\Phi_v$ by $\alpha_v$ and $\beta_v$. Denote the eigenvalues of $\Phi_{\tau_2 v}$ by $\gamma_{\tau_2 v}$ and $\delta_{\tau_2 v}$. Over $L$, $\Phi_w$ acts via $\rho_l^T|L$ with eigenvalues $\alpha_v^{f_1} = \alpha_w$ and $\beta_v^{f_1} = \beta_w$. Likewise, $\Phi_{\overline{\tau_2}w}$ acts with eigenvalues $\gamma_{\tau_2 v}^{f_2} = \gamma_{\overline{\tau_2}w}$ and $\delta_{\tau_2 w}^{f_2} = \delta_{\overline{\tau_2}w}$. Hence $\Phi_w$ acts via ${}^{[\overline{\tau_2}]}\rho_l^T|L$ with eigenvalues $\gamma_{\overline{\tau_2}w}$ and $\delta_{\overline{\tau_2}w}$. Note the product relation $\gamma_{\overline{\tau_2}w}\delta_{\overline{\tau_2}w} = \zeta q_{\overline{\tau_2}w} = \zeta q_w$, where $\zeta$ is a root of unity, which is obvious since, by the global Langlands correspondence, $det(\rho_l^T(\Phi_v)) = \mu q_v$ with a root of unity $\mu$, where $q_v$, $q_w$, and $q_{\overline{\tau}_2 w}$ are the numbers of elements of the residue fields associated to $v$, $w$, and $\overline{\tau}_2 w$.

By definition of $R_l(\pi)$, $\Phi_w$ acts via $R_l(\pi)$ with eigenvalues $a = \alpha_w\gamma_{\overline{\tau_2}w}$, $b = \alpha_w\delta_{\overline{\tau_2}w}$, $c = \beta_w\gamma_{\overline{\tau_2}w}$, and $d = \beta_w\delta_{\overline{\tau_2}w}$.

By Proposition 2, $a, b, c$ and $d$ are $q_w$-Weil numbers. Hence $ab = \alpha_w^2\zeta q_w$, and so $\alpha_w^2$ is a $q_w$-Weil number. Since $\alpha_w^2 = (\alpha_v^2)^{f_1}$, we see that $\alpha_v^2$ is a $q_v$-Weil number. Likewise, $\beta_v^2$ is a $q_v$-Weil number. If we let $|\alpha_v^2| = q_v^{l/2}$ and $|\beta_v^2| = q_v^{m/2}$ then $|\alpha_v| = q_v^{l/4}$ and $|\beta_v| = q_v^{m/4}$, for integers $l$ and $m$.

Recall now the following ([Sha]) Ramanujan estimate:

$$q_v^{-1/5} < |\alpha_v|, |\beta_v| < q_v^{1/5},$$

which applies to all unitary cusp forms $\pi'$ for $GL(2)$ over any number field, at an unramified place $v$ for $\pi'$. Since in our case $\pi \otimes |\cdot|^{1/2}$ is unitary, we see that the exponents $q_v^{\frac{l-2}{4}}$ and $q_v^{\frac{m-2}{4}}$ must be compatible with this estimate. Evidently this happens if and only if $l = m = 2$, which is precisely what we needed.

Of course, this result about $\pi$ implies something about $\rho_l^T$:

**Corollary 7.** Let $\pi$ be a Hilbert modular form of type MC. Then any l- adic representation $\rho_l^T$ which satisfies the GLC satsifies WMC at all places $v$ prime to l.

Remark. An easy extension of the above method shows RC at all places for all $F$, at least under the congruence condition on the weights. To prove RC at the place $v$, it is enough to choose any totally real quadratic extension $K$ of $F$. Then, defining B over $K$ as above, one proves RC, by the method here, at each place of $K$ for the base change $\pi_K$ of $\pi$ from $F$ to $K$. But it is easy to see that RC holds for $\pi_K$ at a place $w$ of $K$ iff it holds for $\pi$ at the place of $F$ under $w$. Thus, RC may be proved for all Hilbert modular forms which satisfy the congruence condition at infinity by a uniform method which reduces the problem to the calculation of [Re1] and Shahidi's estimate.

**Proof of the Corollary**. Indeed, it only remained, in view of the work of Carayol ([Ca]), to establish the result at the unramified places of $\pi$, and this is precisely the RC.

### 4.2 Geometric Proof of Theorem 1.

There exists a finite extension $\overline{L}_{\overline{u}}$ of $L_w$ over which the generic fiber $\overline{Sh_{B,W}}$ of a semistable alteration of $Sh_{B,W}$ is defined. Then $H^2(Sh_{B,W}, \overline{\mathbf{Q}_l})$ is direct summand, as $D_{\overline{u}}$-module of $H^2(\overline{Sh_{B,W}}, \overline{\mathbf{Q}_l})$. This latter group satisfies WMC by [RZ]. Now $R_l(\pi)$ is, after some base change, a tensor product, and its associated Weil-Deligne parameter is thus a tensor product as well. We now note the following simple result whose proof is left to the reader:

**Lemma**. Let $V_1$ and $V_2$ be 2-dimensional representations of a Weil-Deligne group $WD$. Suppose that $V_1 \otimes V_2$ satisfies WMC, and suppose that the modules $\Lambda^2(V_i)$ are pure of weight 2. Then each $V_i$ is pure of weight 1.

Applying this with $V_1 = \rho_l^T$, we conclude that $\rho_l^T|_{D_{\overline{u}}}$ is pure of weight 1 at $u$, and hence $\rho_l^T$ is pure of weight 1 at $v$. Hence, since Local-Global Compatibility is known ([T1]), $\pi_v$ satisfies RC.

## 5 Weight-Monodromy Conjecture for Quaternionic Shimura Varieties.

### 5.1 Proposition 8.

Let F be a number field and let V be a variety defined over $F$. Let l be a rational prime with $(v, l) = 1$. Let $v$ be a finite place of $F$. Then if the WMC holds at the place $v$ for the semisimplification of $H^j(X, \overline{\mathbf{Q}_l})$ as a $\Gamma_F$-module, then the WMC holds for $H^j(X, \overline{\mathbf{Q}_l})$ at $v$.

**Proof.** This is just a geometric restatement of a special case of Proposition 3.

### 5.2 Proof of Theorem 2:WMC for Quaternionic Shimura varieties.

Let, with notations as above, $Sh_{B,W}$ be a quaternionic Shimura variety of dimension r. The Hecke algebra at level W acts semisimply, and $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})$ is thus a direct sum of isotypic components $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})$. It is sufficient, in view of Proposition 1, to show that WMC holds for each of these components. If $\pi_f$ is one-dimensional, the result of Reimann ([Re1], Theorem

11.6) shows that after a finite base change to a number field $L$, the character of the Galois action on $H^*(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$ is a sum of powers of the cyclotomic character at almost all, and hence, by Cebotarev, all, finite places. Since $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$ is pure of weight $j$ at almost all finite places $v$, the Galois action on it over $L$ is a multiple of $\chi_l^{-j}$ where $\chi_l$ is the $l$-adic cyclotomic character. Hence $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$ is pure of weight $j$ at all places. (We note that this fact is much less deep: it is not hard to see that all of $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$ for $j \neq r$ is algebraic, generated on a single geometrically connected component by the $j$-fold products of the $r$ Chern classes of the $r$ line bundles defined by the factors of automorphy attached to the non-compact archimedian places. ) Thus, to prove Theorem 3, we need only consider $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$ where $\pi_f$ is infinite dimensional. In this case, $\pi_\infty$ is discrete series and $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W}) \neq 0$ iff $j = r$. Let $C$ be an irreducible subquotient of $H^j(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi_{f,W})$. Then, by Proposition 6, $C$ is a direct summand of $R_l(JL(\pi'))$ for some $\pi'$. Note that each ${}^{[\overline{\tau_j}]}\rho^T_{l,L}$ is semisimple and satisfies the WMC at each finite place, since $\rho^T_l$ has these properties. Since, for $L$ as before, $R_l(JL(\pi')|_L$ is a tensor product of such representations, it also satisfies WMC at each finite place. Hence the summand $C$ satisfies WMC and consequently the semisimplification of $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})$ as $\Gamma_L$-module also satisfies WMC at each finite place. By Proposition 3, this means $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})$ itself satisfies WMC.

### 5.3 Proof of Theorem 3: Langlands' Conjecture

We recall the statement. Let $\pi'$ be a cuspidal holomorphic automorphic representation of $G(\mathbf{A}_F)$ having weight 2 at each unramified infinite place and with central character $|\cdot|^{-1}\Psi$ where $\Psi$ is a character of finite order. Let $F'$ be the canonical field of definition of $Sh_{B,W}$. Then, for each finite place $v$ of $F'$, the element

$$(\rho^*_{W,v}, N_{W,v})$$

of $Rep^s(WD_{F_v})$ defined by the restriction of the $\Gamma_{F'}$ action on $H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_f)$ to a decomposition group for $v$, coincides with the class

$$m(\pi'_f, W) r_B(\sigma(JL(\pi')_p)|_{WD_{F'_v}}),$$

where

1. the non-negative integer $m(\pi'_f, W)$ is defined by

$$dim(H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})) = 2^r m(\pi'_f, W),$$

2. $p$ is the rational prime under $v$ and $JL(\pi')$ is regarded as an automorphic representation of the $\mathbf{Q}$-group $Res_{F/\mathbf{Q}}(GL(2))$ whose Langlands parameter at $p$ is

$$\sigma(JL(\pi'))_p : WD_p \to^L Res_{F/\mathbf{Q}}(GL(2))$$

([BR2], 3.5), and

3.
$$r :^L Res_{F/\mathbf{Q}}(GL(2))|_{F'} \to GL(2^r, \mathbf{C})$$

is the representation defined by Langlands (c.f. [BR2], 5.1, 7.2).

**Proof.** As before, although the statement is local, for each $v$, the proof proceeds via the global Galois representations. In order to see clearly what is being claimed, we review the key definitions.

Let

$$R_l^T = r_B(Ind_{\mathbf{Q}}^F(\rho_l^T(JL(\pi')))|_{\Gamma_{F'}}).$$

Here, for any two dimensional $l$-adic representation $\rho$ of $\Gamma_F$,

$$Ind_{\mathbf{Q}}^F(\rho) : \Gamma_{\mathbf{Q}} \to^L Res_{F/\mathbf{Q}}(GL(2))_l$$

is a representation of $\Gamma_{\mathbf{Q}}$ into the $l$-adic L-group

$$^L Res_{F/\mathbf{Q}}(GL(2))_l.$$

This latter group is defined in general as in [BR2], 3.5 using groups $\hat{G}_l = GL(2, \overline{\mathbf{Q}}_l)$ in lieu of the complex groups $\hat{G} = GL(2, \mathbf{C})$. Thus, in this case,

$$^L Res_{F/\mathbf{Q}}(GL(2))_l = GL(2, \overline{\mathbf{Q}}_l)^{Hom(F,\mathbf{R})} \times \Gamma_{\mathbf{Q}}$$

is the semidirect product defined via the action: if $g = (g_\tau)_{\tau \in Hom(F,\mathbf{R})}$ and $\eta \in \Gamma_{\mathbf{Q}}$, then $\eta(g)_\tau = g_{\eta^{-1}\tau}$ . The homomorphism

$$I = Ind_{\mathbf{Q}}^F(\rho_l^T(R))$$

is defined as

$$I(\eta) = ((\rho_l^T(\eta_{\overline{\tau}})_{\tau \in Hom(F,\mathbf{R})}), \eta)$$

where the $\overline{\tau}$ are a set of representatives in $\Gamma_{\mathbf{Q}}$for the $\tau$, and $\eta_{\overline{\tau}}$ is defined by the identity

$$\eta\overline{\eta^{-1}\tau} = \overline{\tau}\eta_{\overline{\tau}},$$

for all $\eta$ and all $\tau$.

Of course, if $\eta$ fixes the Galois closure of $F$, then

$$\eta_{\overline{\tau}} = \overline{\tau}^{-1}\eta\overline{\tau},$$

so $I$ is expressed in terms of the conjugates ${}^{\overline{\tau}}\rho_l^T$ of $\rho_l^T$.

We denote the inverse image of $\Gamma_{F'} \subseteq \Gamma_{\mathbf{Q}}$ in ${}^L Res_{F/\mathbf{Q}}(GL(2))_l$ by

$$^L Res_{F/\mathbf{Q}}(GL(2))_l|_{\Gamma_{F'}}.$$

On this latter group is defined the irreducible representation $r_B$ on $\overline{\mathbf{Q}_l}^{2^r}$. We review its construction. Recall that $J_{F,nc} = \{\tau_1, ..., \tau_r\}$ is an ordering of the set of real embeddings $J_{F,\mathbf{R}} \subseteq Hom(F, \mathbf{R})$ of $F$ where $B$ is split. Then on the connected component $GL(2, \overline{\mathbf{Q}_l})^{Hom(F,\mathbf{R})}$, and for $g = (g_\tau)_{\tau \in Hom(F,\mathbf{R})}$,

$$r_B(g) = \otimes_{i=1}^{i=r} g_{\tau_i}.$$

By definition of $F'$, an $\eta \in \Gamma_{F'}$ on $Hom(F, \mathbf{R})$ defines a permutation of $J_{F,nc}$. If we define $r_B^\eta$ by the rule,

$$r_B^\eta(g) = r_B(\eta(g)),$$

then $r_B^\eta$ is isomorphic to $r_B$. Let

$$P \subset GL(2, \overline{\mathbf{Q}_l})^{Hom(F,\mathbf{R})}$$

be product of the groups of upper triangular matrices in each factor. Then (i) $\eta(P) = P$ and (ii) $r_B(P)$ fixes a unique line $\Lambda$ in $\overline{\mathbf{Q}_l}^{2^r}$. If $i(\eta)$ is an isomorphism satisfying, for all $g$,

$$i(\eta) r_B(g) = r_B^\eta(g) i(\eta),$$

then $i(\eta)(\Lambda) = \Lambda$. The choice of $i(\eta)$ is, by Schur's Lemma, unique up to a scalar, and we define $r_B(\eta)$ to be the unique choice which leave $L$ pointwise fixed. The rule $r_B((g, \eta)) = r_B(g) r_B(\eta)$ gives the sought representation.

Note that the restriction of $R_l^T$ to the Galois closure $L$ of $F$ is just the representation $R_l(\pi)$ defined in Section 3. Hence $R_l^T$ is semisimple and satisfies WMC at each finite place $v$ of $F'$ where $(v, l) = 1$. Furthermore, for such $v$, the representations $(\rho_v, N_v)$ of $WD_v$ defined by the $\Phi$- semisimplification of the restriction of $R_l^T$ to a decomposition group $D_{\overline{v}}$ at $v$ coincide, since $\rho_l^T$ satisfies the Langlands correspondence, with

$$r_B(\sigma(JL(\pi')_p)|_{WD_{F'_v}}).$$

Now, at this point we know that, for all $v$ prime to $l$, the representations

$$(\rho^*_{W,v}, N_{W,v})$$

of $WD_v$, defined by the restriction to $D_{\overline{v}}$ of the $\Gamma_{F'}$ module

$$H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})$$

satisfy-whatever they may be- WMC. Thus (see 1.12), for each $v$, the nilpotent data $N_v$ and $N_{W,v}$ are uniquely determined by the semisimple representations $\rho_v$ and $\rho^*_{W,v}$. Hence it will suffice to show that

$$m(\pi'_f, W)\rho_v = \rho^*_{W,v}.$$

Now, for almost all $v$, (i) $N_v = 0$ and $N_{W,v} = 0$, (ii) $\rho_v$ and $\rho^*_{W,v}$ are unramified, and (iii) the computation ([Re1],[L1],[BrLa]) of the unramified zeta function shows exactly that this formula holds. Using the $l$-adic Cebotarev theorem again, we see that the semisimplified $\Gamma_{F'}$-module

$$H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})^{ss}$$

is isomorphic to

$$m(\pi'_f, W)R_l^T.$$

Now let $v$ be any place of $F'$ which is prime to l. Then evidently,

$$(H^r(Sh_{B,W}, \overline{\mathbf{Q}_l})(\pi'_{f,W})|D_{\overline{v}})^{ss}$$

is isomorphic to

$$m(\pi'_f, W)((R_l^T)|D_{\overline{v}})^{ss}.$$

Since the former gives rise to the parameter $\rho^*_{W,v}$ and the latter gives rise to $m(\pi'_f, W)\rho_v$, we are done.

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