THE PARABOLIC ANDERSON MODEL

By Wolfgang König[1,2] and Tilman Wolff[2]

Weierstrass Institute Berlin and TU Berlin

5 September, 2011

Abstract. This is a survey on the parabolic Anderson model.

Contents

1. Introduction

Random motions in random media are an important subject in probability theory since there are a lot of applications to real-world problems in the sciences, like astrophysics, magnetohydrodynamics, chemical reactions. For these reasons and also because its mathematical interest, they have been studied a lot since decades, with a particular intensity in the last twenty years. There is a number of different models of ranodm motions in random media, like random walk in random environment, the random conductance model, random walk in random scenery, random walk in a random potential. Becaues of the variety of models, there is also a variety of questions and of mathematical methods to solve the questions, like homogenisation, subadditive ergodic theorems, Lyapounov exponents.

In this survey, we are interested in the model of a random walk in a random potential, often also called the *parabolic Anderson model (PAM)*. Here the walk has a strong tendency to be confined to an extremely preferable part of the random medium. Therefore, the global properties of the system is not determined by an average behaviour (like in situations where homogenisation works well), but by some local extreme behaviour. As a consequence, the rigorous work on the PAM owes much to theory of large deviations, which describes the exponential rates of probabilities of rare events.

The PAM has become a popular model to study among probabilists and mathematical physicists, since there are a lot of interesting an fruitful connections to other interesting topics like branching

[1]Institute for Mathematics, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany, koenig@math.tu-berlin.de

[2]Weierstrass Institute Berlin, Mohrenstr. 39, 10117 Berlin, koenig@wias-berlin.de and wolff@wias-berlin.de

AMS Subject Classification: 60J65, 60J55, 60F10.

Keywords: random walk in random potential, heat equation with random coefficients, Anderson localisation, mass concentration, intermittency, spectral expansion, large deviations.

random walks with random branching rates, the spectrum of random Schrödinger operators, and certain variational problems. The mathematical activity on the PAM is on a high level since 1990, and many specific and deeper questions and variants were studied specially in the last few years, like time-dependent potentials, connections with Anderson localisation or transitions between quenched and annealed behaviour. For this reason, it seems rather appropriate to provide a survey that collects, in a pedagogical manner, most of the relevant investigations and their interrelations, and to put them into a unifying perspective.

These notes partally rely on the older survey [GK05], but takes the freedom to stress new aspects, give much more intuitive comments that might be helpful for the beginner, and of course collects and comments the latest developments of the work on the PAM.

In Section 1.1 we introduce the model and the relevant questions, explain the heuristics and survey the most important tools. One of the most fundamental questions, the asymptotics of the moments of the total mass of the model, is heuristically explained in Section 2: first we reveal the mechanism, then we bring and comment some detailed formulas. The almost sure asymptotics of the total mass is explained in Section 3, again first the mechanism and afterwards explicit formulas. Section 4 is devoted to the question where the total mass mainly comes from, to concentration properties of the model. Here we also explain details from the yet unfinished work [BK11+], which reveals rigorous connections to Anderson localisation. Finally, in Section 5 we briefly summarize some directions that have been recently studied, like acceleration and deceleration, time-dependent potentials and ageing.

1.1 The parabolic Anderson model.

We consider the continuous solution $u\colon [0,\infty)\times\mathbb{Z}^d\to[0,\infty)$ to the *Cauchy problem* for the *heat equation with random coefficients* and localised initial datum,

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\rm d}u(t,z)+\xi(z)u(t,z), \qquad \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d, \tag{1.1}$$

$$v(0,z) = \delta_0(z), \qquad \text{for } z\in\mathbb{Z}^d. \tag{1.2}$$

Here $\xi=(\xi(z)\colon z\in\mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty,\infty)$, and $\Delta^{\rm d}$ is the discrete Laplacian,

$$\Delta^{\rm d}f(z)=\sum_{y\sim z}\big[f(y)-f(z)\big], \qquad \text{for } z\in\mathbb{Z}^d,\ f\colon\mathbb{Z}^d\to\mathbb{R}.$$

The parabolic problem (1.1) is called the *parabolic Anderson model (PAM)*. The almost sure existence and uniqueness of the solution $u(t,\cdot)$ holds [GM90] if the potential satisfies the mild integrability condition $\langle\xi(0)_+^\alpha\rangle<\infty$ for some $\alpha\in(d,\infty)$, where $\langle\cdot\rangle$ denotes expectation with respect to the potential ξ and the index + the positive part.

The PAM describes a random particle flow in $\mathbb{Z}^d$ through a random field of sinks and sources, corresponding to lattice points z with $\xi(z)<0$ and $\xi(z)>0$, respectively.[4] Two competing effects are present: the diffusion mechanism governed by the Laplacian, and the local growth governed by the potential. The diffusion tends to make the random field $u(t,\cdot)$ flat, whereas the random potential ξ has a tendency to make it irregular. There is an interpretation in terms of a branching process in a field of random branching rates, see Remark 1.2. The operator $\Delta^{\rm d}+\xi$ appearing on the right is called the *Anderson Hamiltonian*; its spectral properties are well-studied in mathematical physics, see Remark 1.8.

[4]Sites x with $\xi(x)=-\infty$ may be allowed and interpreted as ('hard') traps or obstacles, sites with $\xi(x)\in(-\infty,0)$ are sometimes called 'soft' traps.

We refer the reader to [M94] and [CM94] for more background, to [GM90] for basic mathematical properties of the model, and to [GK05] for a survey on mathematical results up till 2005. Our presentation partially draws on [GK05], however, we intended for a more pedagogical approach and added all the results of the literature until 2011.

We distinguish between the so-called *quenched* setting, where we consider $u(t,\cdot)$ almost surely with respect to the medium ξ, and the *annealed* one, where we average with respect to ξ. It is clear that the quantitative properties of the solution strongly depend on the distribution of the field ξ (more precisely, as we we will see, on the upper tail of the distribution of the random variable $\xi(0)$), and that different phenomena occur in the quenched and the annealed settings.

Our main purpose is the description of the solution $u(t,\cdot)$ asymptotically as $t\to\infty$. One of the main objects of interest is the *total mass* of the solution,

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad \text{for } t>0. \tag{1.3}$$

We ask the following questions:

(i) What is the asymptotic behavior of $U(t)$ as $t\to\infty$? (in the annealed and in the quenched setting)
(ii) Where does the main mass of $u(t,\cdot)$ stem from? What are the regions that contribute most to $U(t)$? What are these regions determined by? How many of them are there and how far away are they from each other?
(iii) What do the typical shapes of the potential $\xi(\cdot)$ and of the solution $u(t,\cdot)$ look like in these regions?
(iv) What is the behaviour of the entire process $(u(t,\cdot))_{t\in[0,\infty)}$ of the mass flow? Does it exhibit ageing properties?

Remark 1.1. (Intermittency.) The long-time behaviour of the parabolic Anderson problem is well-studied in the mathematics and mathematical physics literature because it is an important example of a model exhibiting an *intermittency effect*. This means, loosely speaking, that most of the total mass $U(t)$ defined in (1.3) is concentrated on a small number of remote islands, called the *intermittent islands*. A manifestation of intermittency in terms of the moments of $U(t)$ is as follows. For $0<p<q$, the main contribution to the q^{th} moment of $U(t)$ comes from islands that contribute only negligibly to the p^{th} moments. Therefore, intermittency can be defined by the requirement,

$$\limsup_{t\to\infty} \frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}} = 0, \qquad \text{for } 0<p<q, \tag{1.4}$$

where $\langle\,\cdot\,\rangle$ denotes expectation with respect to ξ. Whenever ξ is truly random, the parabolic Anderson model is intermittent in this sense, see [GM90, Theorem 3.2]. ◇

1.2 Interpretations and motivations

The PAM has a lot of relations to other questions and models, which explains the great interest that the PAM receives. We briefly survey the most important ones.

Remark 1.2. (Branching process with random branching rates.) The solution u to (1.1) also admits a branching particle dynamics interpretation [GM90]. Imagine that initially, at time $t=0$, there is a single particle at the origin, and all other sites are vacant. This particle moves according to a continuous-time symmetric random walk with generator $\kappa\Delta$. When present at site x, the particle is split into two particles with rate $\xi_+(x)$ and is killed with rate $\xi_-(x)$, where $\xi_+ = (\xi_+(x))_{x\in\mathbb{Z}^d}$ and $\xi_- = (\xi_-(x))_{x\in\mathbb{Z}^d}$ are independent random i.i.d. fields ($\xi_-(x)$ may attain the value ∞). Every particle continues from its birth site in the same way as the parent particle, and their movements are independent. Put $\xi(x) = \xi_+(x) - \xi_-(x)$. Then, given ξ_- and ξ_+, the expected number of particles present at the site x at time t is equal to $u(t,x)$. Here the expectation is taken over the particle motion and over the splitting resp. killing mechanism, but not over the random medium (ξ_-, ξ_+). $\diamond$

Some of the most interesting applications of the PAM are best explained in terms of an explicit formula for the solution in terms of random walks.

Remark 1.3. (Feynman-Kac formula.) A very useful standard tool for the probabilistic investigation of (1.1) is the well-known *Feynman-Kac formula* for the solution u, which (after time-reversal) reads

$$u(t,x) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\delta_x(X(t))\Big], \qquad (t,x)\in[0,\infty)\times\mathbb{Z}^d, \tag{1.5}$$

where $(X(s))_{s\in[0,\infty)}$ is continuous-time random walk on $\mathbb{Z}^d$ with generator Δ starting at $x\in\mathbb{Z}^d$ under $\mathbb{E}_x$.

QUELLE ANGEBEN? ETWAS MEHR ERLAEUTERN?

$\diamond$

Remark 1.4. (Randomly transformed path measure.) The intermittency effect may also be studied from the point of view of *typical paths* $X(s)$, $s\in[0,t]$, giving the main contribution to the expectation in the Feynman-Kac formula (1.5). On the one hand, the random walker X should move quickly and as far as possible through the potential landscape to reach a region of exceptionally high potential and then stay there up to time t. This will make the integral in the exponent on the right of (1.3) large. On the other hand, the probability to reach such a distant potential peak up to t may be rather small. Hence, the first order contribution to $U(t)$ comes from paths that find a good compromise between the high potential values and the far distance. This contribution is given by the height of the peak. The second order contribution to $U(t)$ is determined by the precise manner in which the optimal walker moves within the potential peak, and this depends on the geometric properties of the potential in that peak. $\diamond$

Remark 1.5. (Survival probabilities.) The case when the field ξ assumes the values $-\infty$ and 0 only has a nice interpretation in terms of survival probabilities and is therefore of particular importance. The fundamental papers [DV75] and [DV79] by Donsker and Varadhan on the Wiener sausage contain apparently the first substantial annealed results on the asymptotics for the PAM. In the nineties, the thorough and deep work by Sznitman (see his monograph [S98]), pushed the rigorous understanding of the quenched situation much further.

An important submodel is *Brownian motion in a Poisson field of traps.* We consider the continuous case, i.e., the version of (1.1) with $\mathbb{Z}^d$ replaced by $\mathbb{R}^d$ and the lattice Laplacian Δ^{d} replaced by the usual Laplace operator Δ. The field ξ is given as follows. Let $(x_i)_{i\in I}$ be the points of a homogeneous Poisson point process in $\mathbb{R}^d$, and consider the union $\mathcal{O}$ of the balls $B_a(x_i)$ of radius a around the

Poisson points x_i. We define a random potential by putting

$$\xi(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{O}, \\ -\infty & \text{if } x \in \mathcal{O}. \end{cases} \tag{1.6}$$

The set $\mathcal{O}$ receives the meaning of the set of 'hard traps' or 'obstacles'. Let $T_{\mathcal{O}} = \inf\{t > 0 \colon X(t) \in \mathcal{O}\}$ denote the entrance time into $\mathcal{O}$ for a Brownian motion $(X(t))_{t\in[0,\infty)}$. Then we have the Feynman-Kac representation

$$u(t,x) = \mathbb{P}_0\,(T_{\mathcal{O}} > t,\, X(t) \in \,\mathrm{d}x) \,/\, \mathrm{d}x,$$

i.e., $u(t,x)$ is equal to the sub-probability density of $X(t)$ on survival in the Poisson field of traps by time t for Brownian motion starting from the origin. The total mass $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$ is the survival probability by time t. Here, one is interested in the path measure that arises when conditioning on the event $\{T_{\mathcal{O}} > t\}$, i.e., transforming with the Radon-Nikodym density $\mathbb{1}\{T_{\mathcal{O}} > t\}/U(t)$.

It is easily seen that the first moment of $U(t)$ coincides with a negative exponential moment of the volume of the Wiener sausage $\bigcup_{s\in[0,t]} B_a(X(s))$.

Donsker and Varadhan analyzed the leading asymptotics of $\langle U(t)\rangle$ by using their large deviation principle for Brownian occupation time measures. The relevant islands have radius of order $\alpha(t) = t^{1/(d+2)}$. To handle the quenched asymptotics of $U(t)$, Sznitman developed a coarse-graining scheme for Dirichlet eigenvalues on random subsets of $\mathbb{R}^d$, the so-called method of enlargement of obstacles (MEO). The MEO replaces the eigenvalues in certain complicated subsets of $\mathbb{R}^d$ by those in coarse-grained subsets belonging to a *discrete* class of much smaller combinatorial complexity such that control is kept on the relevant properties of the eigenvalue.

Qualitatively, the considered model falls into the class of bounded from above fields introduced in Section 2.3 with $\gamma = 0$.

The discrete version is *Simple random walk among Bernoulli traps.* Consider the i.i.d. field $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ where $\xi(x)$ takes the values 0 or $-\infty$ only. Again, $u(t,x)$ is the survival probability of continuous-time random walk paths from 0 to x among the set of traps $\mathcal{O} = \{y \in \mathbb{Z}^d \colon \xi(y) = -\infty\}$.

In their paper [DV79], Donsker and Varadhan also investigated the discrete case and described the logarithmic asymptotics of $\langle U(t)\rangle$ by proving and exploiting a large deviation principle for occupation times of random walks. Later Bolthausen [B94] carried out a deeper analysis of $\langle U(t)\rangle$ in the two-dimensional case using refined large deviation arguments. Antal [A94], [A95] developed a discrete variant of the MEO and demonstrated its value by proving limit theorems for the survival probability $U(t)$ and its moments. $\diamond$

Remark 1.6. (Other types of potentials.) $\diamond$

1.3 Relation with spectral properties

It is common in statistical mechanics that the analysis of the partition sum of a model (here the total mass $U(t)$) is intimately connected with the analysis of the transformed path measure (here the solution $u(t,\cdot)$), i.e., of the relevant regions and the mechanism that makes them relevant. The first observation is that, via the Feynman-Kac formula in (1.5), one sees that $U(t)$ is equal to the t-th positive exponential moment of the quantity

$$Y_t = \frac{1}{t}\int_0^t \xi(X_s)\,\mathrm{d}s,$$

the average of the potential values along the random walk path[5]. Hence, the limiting behaviour of $U(t)$ as $t \to \infty$ is equivalent to the maximisation of Y_t.[6] Hence, the question arises what behaviour of the path and of the potential maximises this quantity.

Certainly, an optimisation of Y_t should be achieved by making the potential values extremely large in a suitable area and to confine the random walk path $X_{[0,t]}$ to that area. That area must be chosen with a compromise, as large areas will be costly for the potential, and small areas will make it expensive for the path to stay inside this area for a long time. The probabilistic cost for the potential is easily qunatified in terms of the upper tails of its distribution; here an appropriate assumption will make this explicit. The question arises how the probabilistic cost for the random walk can be quantified. This will effectively be done in terms of spectral properties as follows. It is known that the probability that a continuous-time random walk stays in a given bounded set $B \subset \mathbb{Z}^d$ by time t is, on a logarithmic scale, equal to $\mathrm{e}^{-t\lambda_1(B)}$, where $\lambda_1(B)$ is the principal (i.e., largest) eigenvalue of the generator of the walk, Δ^{d}, with Dirichlet (i.e., zero) boundary condition. This assertion can be proved in terms of a large-deviation principle [DV75], [G77], but also follows from a spectral decomposition. Let us introduce this important tool now.

Remark 1.7. (Eigenvalue expansion.) The large-time behavior of U is determined by the spectral properties of the *Anderson Hamiltonian*

$$\mathcal{H} = \Delta^{\mathrm{d}} + \xi, \tag{1.7}$$

which is the operator that appears on the right-hand side of the parabolic Anderson problem in (1.1).[7] We introduce Dirichlet (i.e., zero) boundary condition in a large centred box B and consider the solution, u_B, of (1.1) in B. The solution u_B admits the spectral representation

$$u_B(t, \cdot) = \sum_{k=1}^{|B|} \mathrm{e}^{t\lambda_k} v_k(0) v_k(\cdot) \tag{1.8}$$

with respect to the random eigenvalues λ_k and an orthonormal basis consisting of corresponding random eigenfunctions v_k, both depending also on B. QUELLE ANGEBEN? We always pick the eigenvalues $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots$ in decreasing order and the principal eigenfunction v_1 positive. The eigenvalue expansion (sometimes called *Fourier expansion* or *spectral decomposition*) is an important tool for proofs and yields an instructive explanation of the asymptotics from a spectral point of view, see also Remark 1.8. $\diamond$

Remark 1.8. (Anderson localisation.) We now consider the spectral representation (1.8) in view of asymptotics as $t \to \infty$. Let us pick $B = B_t$ so large that $u_{B_t}(t, \cdot)$ is a very good approximation of $u(t, \cdot)$. (E.g., for potentials ξ that possess all positive exponential moments, it is enough to pick the diameter of B roughly of the order $t \log^2 t$.) It is the main assertion of the famous *Anderson localisation prediction* that, at least at the boundary of the spectrum of $\mathcal{H}$ (i.e., for small k), the eigenfunctions v_k should be exponentially localised in centres x_k that are far away from each other.

[5]This quantity tY_t is sometimes called *random walk in a random scenery* at continues to be studied for discrete-time simple random walk.

[6]This is in line with the well-known fact that $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\mathrm{e}^{tY}] = \operatorname{esssup} Y \in (-\infty, \infty]$ for any random variable Y.

[7]Note that we do not put a minus sign in front of the Laplace operator, unlike the mathematical physics community does. In particular, we do not speak of the 'bottom of the spectrum' but of the 'top', and 'deep valleys' of the potential are here 'high exceedances'

Hence, the leading eigenvalues λ_k and the corresponding eigenfunction v_k resemble the local *principal* eigenvalue/function pair of $\mathcal{H}$ in a neighbourhood of X_K.

In the limit $t \to \infty$, we can neglect all the summands in (1.8) with large k, because of the appearance of the exponential term $e^{t\lambda_k}$. In the remaining sum on finitely many k's, only those summands k contribute to (1.8) for which the term $e^{t\lambda_k}v_k(0)$ is maximal (assuming, for simplicity, that $v_k(0)$ is nonnegative). But $v_k(0)$ is the smaller, the farther away from zero the localisation centre x_k is. Hence, there is a optimisation mechanism at work; the best compromise between large eigenvalue and proximity of the main bulk of the eigenfunction mass determines the large-t asymptotics of $u(t,\cdot)$. (This is the spectral counterpart of the probabilistic mechanism explained in Remark (1.4).) Hence, $u(t,\cdot)$ indeed looks like a weighted superposition of high peaks concentrated on distant islands. Furthermore, one can expect that, in many cases, the total mass $U(t)$ is asymptotically concentrated in just one peak.

The theory of Anderson localisation continues to be one great source of interest in the PAM. This theory is interested in eigenfunction concentration properties in all parts of the spectrum of $\mathcal{H}$, not only at the edge of the spectrum, as the PAM theory is. With the help of the spectral properties of $\mathcal{H}$, one wants to describe electrical conductance properties of alloys of metals or optical properties of glasses with random impurities. Therefore, one is naturally mainly interested in *bounded* potentials here, in contrast with the PAM theory, where unbounded potentials are of prominent interest. See [K08] for an extended survey on Anderson localisation and further reading. ◇

Remark 1.9. (The difference between moment asymptotics and almost sure asymptotics.) As we will see in Sections 2 and 3 below, the arguments that explain the asymptotics of the moments of $U(t)$ and those that explain its almost sure asymptotics are quite different (but related). The phenemonological difference between the two is the following. From the Feynman-Kac formula in (1.5) we see that the moments of $U(t)$ are the joint expectations over the path and over the potential. Hence, both random objects can 'work together' according to a joint strategy that is a compromise between the two; both give a contribution that is exponentially costly: the potential assumes high values in a suitable area, the path does not leave that area. In particular, it will be convenient to choose this area centred at the origin, the starting point of the walk, and to pick it equal to one big ball, instead of many small ones that are widely spread (as in the heuristics on the Anderson localisation.

In contrast, in the almost sure setting, the potential makes no particular effort to achieve a particular contirbution, and the random path (here the expectation is taken exclusively) has to cope with that and must 'make the best' out of it. A closer analysis of the potential landscape reveals that it produces, in a large macrobox, many smaller islands on which it is particularly preferable for the random walk, i.e., possesses a large local eigenvalue. This picture will be used to provide lower estimates for $U(t)$, but proofs for the corresponding upper bound will be more technical. ◇

1.4 Summary

In much of the following, we will be working under the assumption that all positive exponential moments of $\xi(0)$ are finite and that the upper tails of $\xi(0)$ possess some mild regularity property. One of the main results of [HKM06] is that four different universality classes of long-time behaviours of the parabolic Anderson model can be distinguished: the so-called double-exponential distribution and some degenerate version of it studied by Gärtner, Molchanov and König [GM98], [GKM07], bounded from above potentials studied by Biskup and König [BK01], and so-called almost bounded potentials studied by van der Hofstad, König and Mörters [HKM06].

It turns out that there is a universal picture present in the asymptotics of the parabolic Anderson model. Inside the relevant islands, after appropriate vertical shifting and spatial rescaling, the potential ξ turns out to asymptotically approximate a universal, non-random shape, V, which is determined by a characteristic variational problem. The absolute height of the potential peaks and the diameter of the relevant islands are asymptotically determined by the upper tails of the random variable $\xi(0)$, while the number of the islands and their locations are random. Furthermore, after multiplication with an appropriate factor and rescaling, also the solution $u(t,\cdot)$ approaches a universal shape on these islands, namely the principal eigenfunction of the Hamiltonian $\Delta^{\mathrm{d}} + V$ with V the above universal potential shape. Remarkably, there are only four universal classes of potential shapes for the PAM in (1.1), see Section 2.3 for details.

The thicker the upper tails of the single-site distribution is, i.e., the easier it is for the potential to develop extremely high values, the simpler the spatial structure of the solution $u(t,\cdot)$ is. Indeed, in the extreme case of the Pareto distribution with polynomial tails (note that this distribution does not belong to the above classification, since the exponential moments are infinite), it turns out that an extreme concentration takes place: the main contribution to $U(t)$ comes from just one single site, Z_t. It is even possible to describe the entire process $(Z_t)_{t\in[0,\infty)}$ and to prove ageing properties of the PAM. This answers all the four main questions that we formulated above in a particularly satisfactory way. We survey this in Section 4.2 and heuristically explain how to achieve a similarly complete picture in the above mentioned universality classes.

For a general discussion we refer to the monograph by Carmona and Molchanov [CM94], the lectures by Molchanov [M94], and also to the monograph [S98] by Sznitman on the spatially continuous case. A discussion from a physicist's and a chemist's point of view in the particular case of trapping problems, including a survey on related mathematical models and a collection of open problems, is provided in [HW94]. A general mathematical background for the PAM is provided in [GM90].

2. Logarithmic asymptotics for the moments of the total mass

In this section, we explain, on a heuristic level, what the asymptotics of the logarithm of the moments of $U(t)$ are determined by, and how they can be described. We do this under the basic assumption that all positive exponential moments of $\xi(0)$ are finite, in which case all the moments of $U(t)$ are finite. Hence, the *cumulant generating function* of $\xi(0)$ (often called *logarithmic moment generating function*) defined by

$$H(t) = \log\big\langle \mathrm{e}^{t\xi(0)}\big\rangle, \qquad t > 0, \tag{2.1}$$

receives great interest, since its behaviour as $t \to \infty$ describes the potential close to its essential supremum.

For simplicity we restrict ourselves to the first moment. We give first a heuristic derivation on base of the eigenvalue expansion in (1.8) in Section 2.1, then a second derivation in terms of a large-deviation statement for the local times of the random walk in Section 2.2 and formulate the outcome of these heuristics in Section 2.3. There it turns out that only four different regimes are to be distinguished, and we will provide explicit formulas for the logarithmic asymptotics.

2.1 Heuristics via eigenvalues

The first observation is that, as a consequence of the spectral representation (1.8),

$$U(t) \approx \mathrm{e}^{t\lambda_t(\xi)} \tag{2.2}$$

(in the sense of logarithmic equivalence), where $\lambda_t(\varphi)$ denotes the principal (i.e., largest) eigenvalue of the Anderson operator $\mathcal{H} = \Delta + \varphi$ with zero boundary condition in the 'macrobox' $B_t = [-t,t]^d \cap \mathbb{Z}^d$. Hence, we have to understand the logarithmic asymptotics of high exponential moments of the principal eigenvalue of $\mathcal{H}$ in a large, time-dependent box.

Remark 2.1. (p-th moments.) It is already heuristically clear from (2.2) (and true in all known cases) that the p-th moments of $U(t)$ should have the same asymptotics as the first moments of $U(pt)$.$\diamond$

As we indicated in Remark 1.8, the main contribution to $\langle \mathrm{e}^{t\lambda_t(\xi)} \rangle$ comes from realizations of the potential ξ having high peaks on distant islands in B_t whose radius is of some order $\alpha(t)$ that is much smaller than t. But this implies that $\lambda_t(\xi)$ is close to the principal eigenvalue of $\mathcal{H}$ on one of these islands. Therefore, since the number of subboxes of B_t of radius of order $\alpha(t)$ is negligible on an exponential scale, and since the distribution of ξ is spatially homogeneous, we may expect that

$$\left\langle \mathrm{e}^{t\lambda_t(\xi)} \right\rangle \approx \left\langle \mathrm{e}^{t\lambda_{R\alpha(t)}(\xi)} \right\rangle, \qquad t \to \infty,$$

and the auxiliary parameter R must be picked large in the end.

Remark 2.2. (Choice of $\boldsymbol{\alpha(t)}$.) The choice of the scale function $\alpha(t)$ depends on asymptotic 'stiffness' properties of the potential, more precisely of its tails at its essential supremum, and is determined by a large deviation principle, see (2.9) below. More precisely, we are working here under the following supposition.

Assumption (J). There is an auxiliary scale function η and a non-trivial shape function J such that

$$\lim_{t\to\infty} \frac{1}{\eta(t)} \log \mathrm{Prob}\Big(\xi(0) > \frac{H(t)}{t} + \frac{\eta(t)}{t} x\Big) = -J(x), \qquad x \in \mathbb{R}.$$

Then we define the scale function $\alpha(t)$ by

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,. \tag{2.3}$$

In Section 2.3 we shall see examples of potentials such that $\alpha(t)$ tends to 0, to ∞, or stays bounded and bounded away from zero as $t \to \infty$. In the present heuristics, we assume that $\alpha(t) \to \infty$, which implies the necessity of a spatial rescaling. In particular, after rescaling, the main quantities and objects will be described in terms of the continuous counterparts of the discrete objects we started with, i.e., instead of the discrete Laplacian, the continuous Laplace operator appears etc. The following heuristics can also be read in the case where $\alpha(t) \equiv 1$ by keeping the discrete versions for the limiting objects. $\diamond$

The height of the maximal eigenvalue in B_t can be approximated in terms of the cumulant generating function $H(t)$. Indeed, recall that $\lim_{s\to\infty} H(s)/s$ is equal to the essential supremum of $\xi(0)$. Furthermore, some elementary argument on the maximum of i.i.d. random variables shows that the maximum of the $t^d/\alpha(t)^d$ independent copies of a local eigenvalue $\lambda_{R\alpha(t)}(\xi)$ should be of the order $H(t/\alpha(t)^d)/(t/\alpha(t)^d)$.

The optimal behavior of the field ξ in the 'microbox' $B_{R\alpha(t)}$ is to approximate a certain (deterministic) shape φ after appropriate spatial scaling and vertical shifting. Together with Brownian scaling, this leads to the ansatz

$$\overline{\xi}_t(\cdot) = \alpha(t)^2\Big[\xi\big(\lfloor \cdot\,\alpha(t)\rfloor\big) - \frac{H(t/\alpha(t)^d)}{t/\alpha(t)^d}\Big], \tag{2.4}$$

for the spatially rescaled and vertically shifted potential in the cube $Q_R = (-R, R)^d$. Now the idea is that the main contribution to $\langle U(t)\rangle$ comes from fields that are shaped in such a way that $\overline{\xi}_t \approx \varphi$ in Q_R, for some $\varphi\colon Q_R \to \mathbb{R}$, which has to be chosen optimally. Observe that

$$\overline{\xi}_t \approx \varphi \quad \text{in } Q_R \qquad \Longleftrightarrow \qquad \xi(\cdot) \approx \frac{H(t/\alpha(t)^d)}{t\alpha(t)^d} + \tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big) \quad \text{in } B_{R\alpha(t)}. \tag{2.5}$$

Let us calculate the contribution to $\langle U(t)\rangle$ coming from such fields. Using (2.2), we obtain

$$\big\langle U(t)\,\mathbb{1}\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}\big\rangle \approx \mathrm{e}^{\alpha(t)^d H(t\alpha(t)^d)} \exp\Big\{t\lambda_{R\alpha(t)}\big(\tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\big)\Big\}\,\mathrm{Prob}\big(\overline{\xi}_t \approx \varphi \text{ in } Q_R\big). \tag{2.6}$$

The asymptotic scaling properties of the discrete Laplacian, Δ^{d}, imply that

$$\lambda_{R\alpha(t)}\big(\tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\big) \approx \frac{1}{\alpha(t)^2}\lambda_R^{(\mathrm{c})}(\varphi), \tag{2.7}$$

where $\lambda_R^{(\mathrm{c})}(\varphi)$ denotes the principal eigenvalue of $\Delta + \varphi$ in the cube Q_R with zero boundary condition, and Δ is the usual 'continuous' Laplacian. This leads to

$$\big\langle U(t)\,\mathbb{1}\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}\big\rangle \approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{\frac{t}{\alpha(t)^2}\lambda_R^{(\mathrm{c})}(\varphi)\Big\}\,\mathrm{Prob}\big(\overline{\xi}_t \approx \varphi \text{ in } Q_R\big). \tag{2.8}$$

In order to achieve a balance between the second and the third factor on the right, it is necessary that the logarithmic decay rate of the considered probability is $t/\alpha(t)^2$. Indeed, we have a large deviation principle for the shifted, rescaled field, which reads

$$\mathrm{Prob}\big(\overline{\xi}_t \approx \varphi \text{ in } Q_R\big) \approx \exp\Big\{-\frac{t}{\alpha(t)^2} I_R(\varphi)\Big\}, \tag{2.9}$$

where

$$I_R(\varphi) = \int_{Q_R} J(\varphi(y))\,\mathrm{d}y, \tag{2.10}$$

where J was introduced in Assumption (J) above. (It is not difficult to heuristically derive (2.9) from Assumption (J), approximating the event $\{\overline{\xi}_t \approx \varphi\}$ with $\{\overline{\xi}_t > \varphi\}$.)

Now substitute (2.9) into (2.8). Then the Laplace method tells us that the exponential asymptotics of $\langle U(t)\rangle$ is equal to the one of $\big\langle U(t)\,\mathbb{1}\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}\big\rangle$ with optimal φ. Hence, optimizing on φ and remembering that R is large, we arrive at

$$\big\langle U(t)\big\rangle \approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)} \exp\Big\{-\frac{t}{\alpha(t)^2}\chi\Big\}, \tag{2.11}$$

where the constant χ is given in terms of the characteristic variational problem

$$\chi = \lim_{R\to\infty} \inf_{\varphi\colon Q_R\to\mathbb{R}} \big[I_R(\varphi) - \lambda_R^{(\mathrm{c})}(\varphi)\big]. \tag{2.12}$$

The first term on the right of (2.11) is determined by the absolute height of the typical realizations of the potential and the second contains information about the shape of the potential close to its maximum in spectral terms of the Anderson Hamiltonian $\mathcal{H}$ in this region. More precisely, those realizations of ξ with $\overline{\xi}_t \approx \varphi_*$ in Q_R for large R and φ_* a minimizer in the variational formula in

(2.12) contribute most to $\langle U(t)\rangle$. In particular, the geometry of the relevant potential peaks is hidden via χ in the second asymptotic term of $\langle U(t)\rangle$.

Remark 2.3. (Technical difficulties.) It is technically rather difficult to turn the above heuristics into a rigorous proof, since the properties of the potential and its large-deviation rate function and of the map $\varphi\mapsto\lambda^{(c)}(\varphi)$ in view of continuity, boundedness and compactness are not given to a sufficient extent. More precisely, the large-deviation principle alluded to in (2.9) can *a priori* be proved only in a rather weak topology, the restriction to a box of diameter of order $\alpha(t)$ needs a careful argument, and the map $\varphi\mapsto\lambda^{(c)}(\varphi)$ is neither bounded nor continuous. For these reasons, hardly any proof in the literature follows the above outline. An exception is Sznitman's [S98] *method of enlargement of obstacles* that works in the spatially continuous setting of a Brownian motion in a Poisson field of obstacles. This method is a sophisticated discretisation method which carefully simplifies the shape of the potential ξ according to a precise view at the local and global spectral properties of $\Delta+\xi$. $\diamond$

Remark 2.4. (Potential confinement properties.) The above heuristics suggests, in the spirit of large-deviation theory, that the main contribution to the moments should come from those realizsations of the potential ξ such that the rescaled shifted version $\overline{\xi}_t$ resmbles the members of the set $\mathcal{M}$ of minimizer(s) of the variational formula in (2.12). In other words, there should be a law of large numbers for $\overline{\xi}_t$ in the sense that the event $\{\xi\colon\overline{\xi}_t\notin U(\mathcal{M})\}$, where $U(\mathcal{M})$ is some neighbourhood of $\mathcal{M}$ in a suitable topology. There is no doubt that such a law of large numbers should be valid in great generality, but there are only few proofs for this in the literature. A technical problem may be that the problem in (2.12), and hence also $\mathcal{M}$, is spatially shift-invariant, i.e., one has to cope with the event that $\overline{\xi}_t$ does not resemble any shift of the minimiser(s). Such a statement has been proved in the case of an almost bounded potential ξ (in the notation introduced in Section 2.3 below) in [GK09].$\diamond$

2.2 Heuristics via local times

In this section, we present another route along which the asymptotics of the moments of $U(t)$ can be identified. This route is in a sense 'dual' to the route that we described in Section 2.1: Instead of carrying out first the expectation with respect to the random walk in the Feynman-Kac formula in (1.5) and analysing the ξ-expectation of the resulting expression in the eigenvalue expansion, we now carry out first the ξ-expectation in the Feynman-Kac formula and analyse the resulting expectation over the random walk.

Indeed, we introduce one main object in the probabilistic treatment of the PAM. The *local times* of the random walk $(X_s)_{s\in[0,\infty)}$ measure the time the walker spends in a given lattice point. They are given by

$$\ell_t(z)=\int_0^t\delta_z(X_s)\,\mathrm{d}s,\qquad t>0,z\in\mathbb{Z}^d. \tag{2.13}$$

Then, using that the potential is i.i.d., we easily derive from the definition of $H(t)$ that

$$\langle U(t)\rangle=\Big\langle\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,\mathrm{d}s\Big\}\Big]\Big\rangle=\mathbb{E}\Big[\exp\Big\{\sum_{z\in\mathbb{Z}^d}H(\ell_t(z))\Big\}\Big]. \tag{2.14}$$

We are going to work under the following supposition.

Assumption (Ĥ): *There are a function* $\widehat{H}\colon (0,\infty) \to \mathbb{R}$ *and a continuous auxiliary function* $\eta\colon (0,\infty)\to(0,\infty)$ *such that*

$$\lim_{t\uparrow\infty} \frac{H(ty)-yH(t)}{\eta(t)} = \widehat{H}(y) \neq 0 \qquad \text{for } y\neq 1. \tag{2.15}$$

Assumption (Ĥ) is crucial and will be discussed at length in Section 2.3 below. Let us already remarkt that the function η coincides with the one of Assumption (J) in Remark 2.2. We define the scale function $\alpha(t)$ as in Section 2 by

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,.$$

Let us again assume that $\alpha(t)\to\infty$ for these heuristics. Then we need to consider the spatially rescaled version of the local times,

$$L_t(y) = \frac{\alpha(t)^d}{t}\ell_t(\lfloor \alpha_t y\rfloor), \qquad y\in Q_R. \tag{2.16}$$

Then L_t is a random, L^1-normalised step function. We continue (2.14) with

$$\langle U(t)\rangle = \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\mathbb{E}\Big[\exp\Big\{\eta(t/\alpha(t)^d)\sum_{z\in\mathbb{Z}^d}\frac{H(L_t(z/\alpha(t))t/\alpha(t)^d)-L_t(z/\alpha(t))H(t/\alpha(t)^d)}{\eta(t/\alpha(t)^d)}\Big\}\Big].$$

According to Assumption (Ĥ), we may asymptotically replace the quotient after the sum on z by $\widehat{H}(L_t(z/\alpha(t)))$, and by definition of α, we may replace the prefactor in the exponent by $t/\alpha(t)^{d+2}$. Reducing the sum on $z\in\mathbb{Z}^d$ to a sum on $z\in B_{R\alpha(t)}$ and turning this sum into an integral using the substitution $z=\lfloor y\alpha(t)\rfloor$, we arrive at

$$\langle U(t)\rangle \approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\mathbb{E}\Big[\exp\Big\{\frac{t}{\alpha(t)^2}\int_{Q_R}\widehat{H}(L_t(y))\,\mathrm{d}y\Big\}\Big]. \tag{2.17}$$

A crucial fact is that $(L_t)_{t\in(0,\infty)}$ satisfies a large-deviation principle with speed $t/\alpha(t)^2$ and rate function $g^2\mapsto \|\nabla g\|_2^2$, that is,

$$\mathbb{P}_0(L_t(\cdot)\approx g^2(\cdot) \text{ in } Q_R) \approx \exp\Big\{-\frac{t}{\alpha(t)^2}\|\nabla g\|_2^2\Big\}, \tag{2.18}$$

for any L^2-normalised function $g\in H^1(\mathbb{R}^d)$ with support in Q_R. Now we formally apply Varadhan's lemma to the expectation on the right-hand side of (2.19) and obtain, after letting $R\to\infty$,

$$\langle U(t)\rangle \approx \mathrm{e}^{\alpha(t)^d H(t/\alpha(t)^d)}\exp\Big\{-\frac{t}{\alpha(t)^2}\widetilde{\chi}\Big\}, \tag{2.19}$$

where $\widetilde{\chi}$ is given as

$$\widetilde{\chi} = \inf\Big\{\|\nabla g\|_2^2 - \int_{\mathbb{R}^d}\widehat{H}(g^2(y))\,\mathrm{d}y\colon g\in H^1(\mathbb{R}^d), \|g\|_2=1\Big\}. \tag{2.20}$$

Remark 2.5. ($\widetilde{\chi}=\chi$.) Certainly, (2.11) and (2.19) must coincide; in particular the two variational formulas in (2.12) and (2.20) must be identical. This can also be seen in an analytical way as follows. First recall the well-known Rayleigh-Ritz principle for the principal eigenvalue of $\Delta+\varphi$ in the box Q_R with zero boundary condition,

$$\lambda_R^{(\mathrm{c})}(\varphi) = \sup_{g\in H^1(\mathbb{R}^d), \|g\|_2=1, \mathrm{supp}\,(g)\subset Q_R}\big[\langle g^2,\varphi\rangle - \|\nabla g\|_2^2\big], \tag{2.21}$$

where we wrote $\langle \cdot, \cdot \rangle$ for the standard L^2-inner product. Furthermore, it is not too difficult to identify the function J introduced in Assumption (J) as the Legndre transform of the function $-\widehat{H}$ introduced in Assumption ($\hat{\text{H}}$), that is,

$$J(x) = \sup_y \big(xy + \widehat{H}(y)\big), \qquad x \in \mathbb{R}. \tag{2.22}$$

Combining (2.21) and (2.22), it is straightforward to see that $\widetilde{\chi} = \chi$. ◇

Remark 2.6. (Technical difficulties.) Like for the approach described in Section 2.1, there are a number of technical obstacles that arise when the above heuristics is turned into a rigorous proof. However, since the large-deviations theory for local times of random walks is well-developed, there are techniques that were used for this purpose in the literature. Indeed, most of the available rigorous results run along the above lines and overcome the notorious difficulties. These are similar to the other approach: (1) restriction to a box of radius of order $\alpha(t)$ (this is sometimes called *compactification*), (2) overcoming the lack of boundedness and (3) the lack of continuity of the exponential functional of the local times. Depending on the regime of potential distributions under consideration, the removal of these difficulties is more or less cumbersome.

The 'compactification' is usually done with the help of an elementary periodisation technique, which apparantly has been employed for the first time in related models in [DV75]. Note first that the lower bound is not the problem, since one can introduce zero boundary condition in the box $B_{R\alpha(t)}$ by inserting an indicator on the event that the random walk in the Feynman-Kac formula does not leave that box by time t. It is the upper bound that needs an argument. The simple idea is that the contribution to $\langle U(t)\rangle$ (or, equivalently, to the exponential functional in (**??**)) is larger for the periodised random walk on the box B_R than for the free random walk. (This property is easily seen with the help of convexity.) Applying this to R replaced by $R\alpha(t)$, we see that a transition from the free walk to the periodised walk yields an upper bound. From this point on, one works with periodised boundary condition and derives a variant of the variational formula in (2.20) that is easily seen to converge to χ as $R \to \infty$.

Another technique of 'compactification' is introduced in [GK00] in the continuous setting and was transferred to the discrete setting in [BK01]. The idea is to show that the principal Dirichlet eigenvalue of $\Delta + \varphi$ in some (arbitrarily large) box B is not larger than the maximal Dirichlet eigenvalue of $\Delta + \varphi$ in much smaller, mutually overlapping subboxes of B, subject to a controllable error. When the diameter of the subboxes is chosen as $R\alpha(t)$, then one has reduced the problem to the appropriate size with zero boundary condition, and the error vanishes in the limit $t \to \infty$, followed by $R \to \infty$.

Difficulty (2) is usually handled by some cutting technique, which requires serious work on a case-by-case basis, see e.g. [HKM06]. Difficulty (3) is often taken care of by some smoothing procedure, i.e., by a replacement of the rescaled local times with the convolution with a smooth approximation of the Dirac measure, see e.g. [HKM06]. These two diffiulties did not appear in [GM98] because of the discrete structure: After the 'compactification', the resulting expectation was concntrated on some *finite* subset of $\mathbb{Z}^d$, and boundedness and continuity were present for free.

A quite sophisticated technique for overcoming the lack of boundedness and continuity was exemplified in [HKM06, Section 5] and is based on an explicit upper bound for the joint density of the family of local times for random walks on finite state spaces. This bound was derived in [BHK07] and makes it possible to derive an LDP for the (rescaled) local times even in the *strong* topology, i.e., without any assumption on boundedness or continuity. A drawback of this strategy is an error term that makes it impossible to apply this to boxed that are too large, but in the setting of [HKM06] it worked very satisfactorily. ◇

Remark 2.7. (Path confinement properties.) Analogously to Remark 2.4, it is also tempting to guess that the rescaled local times should satisfy a law of large numbers, i.e., should converge to the minimiser(s) in (2.20). This property has been called the *Brownian confinement property* and was indeed proved in some of the most interesting cases, see [S91] for $d = 2$ and [P99] for $d \geq 3$ in the case of Brownian motion among Poisson obstacles. There, it was not only proved that the convergence takes place towards the set of all minimisers (which contains all spatial shifts of any minimiser), but only to the unique one that is centred, a phenomenon that is often called *localisation*. Even more, a rescaled version of the Brownian motion in the Feynman-Kac formula is shown to converge towards a transformed version of the Brownian motion, the Brownian motion that does not leave a certain ball forever. (On the technical side, such a technique played an important rôle in [GH99] for deriving particularly precise estimates.) In the spatially discrete case (i.e., simple random walk among Bernoulli traps), an analogous result in $d = 2$ was derived in [B94].

There is an analytic difficulty that has to be handled before one can apply probabilistic estimates: One must know that the minimisers of (2.20) are approachable in the topology in which one wants to prove the law of large numbers for the rescaled local times. In other words, one needs a statement like 'if a sequence of admissible functions is such that its values of the target function approach the minimum, then (at least along a subsecquence) this sequence converges to some shift of the minimiser (in the topology that one would like to work with probabilistically).' Proving such a statement is by far not trivial and must be done on a case-by-case basis. ◇

2.3 Four asymptotic regimes

In this section we explain that, under some mild regularity assumptions on the tails of $\xi(0)$ at its essential supremum, there are only four regimes (called universality classes in [HKM06]) of asymptotic behaviors of the PAM. These regimes differ from each other in the order of the size of the relevant islands, in the explicit form of the rate function for the potential and in properties of the minimizers (e.g. compactness/unboundedness of support). The theory of regular functions straightly implies from the main assumption that the main asymptotic quantities in terms of which the asymptotics of the moments are given take only two different and explicit forms. Depending on unboundedly growing or vanishing diameters of the relevant islands, the asymptotic shape is continous (after spatial rescaling) or discrete or even trivially concentrated in just one site. We follow [HKM06].

Our basic assumption on the upper tails of the potential ξ is formulated in terms of the logarithmic moment generating function H in (2.1) as follows.

Assumption ($\hat{\mathbf{H}}$): *There are a function* $\widehat{H}\colon (0,\infty) \to \mathbb{R}$ *and a continuous auxiliary function* $\eta\colon (0,\infty) \to (0,\infty)$ *such that*

$$\lim_{t\uparrow\infty} \frac{H(ty) - yH(t)}{\eta(t)} = \widehat{H}(y) \neq 0 \qquad \text{for } y \neq 1. \tag{2.23}$$

The function $\widehat{H}$ extracts the asymptotic scaling properties of the cumulant generating function H. In the language of the theory of regular functions, the assumption is that the logarithmic moment generating function H is in the de Haan class, which does not leave many possibilities for $\widehat{H}$:

Proposition 2.8. *Suppose that Assumption ($\hat{H}$) holds.*

(i) *There is a $\gamma \geq 0$ such that $\lim_{t\uparrow\infty} \eta(yt)/\eta(t) = y^\gamma$ for any $y > 0$, i.e., η is regularly varying of index γ. In particular, $\eta(t) = t^{\gamma+o(1)}$ as $t \to \infty$.*

(ii) *There exists a parameter $\rho > 0$ such that, for every $y > 0$,*
   (a) $\widehat{H}(y) = \rho \dfrac{y - y^\gamma}{1-\gamma}$ *if $\gamma \neq 1$,*
   (b) $\widehat{H}(y) = \rho y \log y$ *if $\gamma = 1$.*

Our second regularity assumption is a mild supposition on the auxiliary function η. This assumption is necessary only in the case $\gamma = 1$ (which will turn out to be the critical case).

Assumption (K): *The limit $\eta_* = \lim_{t\to\infty} \eta(t)/t \in [0,\infty]$ exists.*

We now introduce a scale function $\alpha\colon (0,\infty) \to (0,\infty)$, by

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,. \tag{2.24}$$

The function $\alpha(t)$ turns out to be the annealed scale function for the radius of the relevant islands in the parabolic Anderson model, this is the function that appeared in the heuristics described in Sections 2.1 and 2.2. We can easily say something about the asymptotics of $\alpha(t)$:

Lemma 2.9. *Suppose that Assumptions ($\hat{H}$) and (K) hold. If $\gamma \leq 1$ and $\eta_* < \infty$, then there exists a unique solution $\alpha\colon (0,\infty) \to (0,\infty)$ to (2.24), and it satisfies $\lim_{t\to\infty} t\alpha(t)^{-d} = \infty$. Moreover,*

(i) *If $\gamma = 1$ and $0 < \eta_* < \infty$, then $\lim_{t\to\infty} \alpha(t) = 1/\sqrt{\eta_*} \in (0,\infty)$.*
(ii) *If $\gamma = 1$ and $\eta_* = 0$, then $\alpha(t) = t^{\nu+o(1)}$ as $t \to \infty$, where $\nu = (1-\gamma)/(d+2-d\gamma)$.*

Now, under Assumptions ($\hat{\text{H}}$) and (K), we can formulate a complete distinction of the PAM into four cases:

(SP) $\eta_* = \infty$ (in particular, $\gamma \geq 1$), the *single-peak case.*
This is the boundary case $\varrho = \infty$ of the double-exponential case. We have $\alpha(t) \to 0$ as $t \to \infty$, as is seen from (2.24), i.e., the relevant islands consist of single lattice sites. As we will see in Section 4.2, this class phenemonologically also contains a number of potentials that have no finite exponential moments.

(DE) $\eta_* \in (0,\infty)$ (in particular, $\gamma = 1$), the *double exponential case.*
This is the case of the double-exponential distribution. By rescaling, one can achieve that $\eta_* = 1$. The parameter ϱ of Proposition 2.8(ii)(b) is identical to the one in (2.26) below. This case is studied in [GM98], [GH99], [GK00], [GKM00], [GKM07], [BK11+] and more papers.

(AB) $\eta_* = 0$ and $\gamma = 1$, *the almost bounded case.*
This is the case of islands of slowly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ slower than any power of t. This case comprises unbounded and bounded from above potentials. This class was introduced in [HKM06] and further studied in [GK09]. It lies in the union of the boundary cases $\varrho \downarrow 0$ of (DE) and $\gamma \uparrow 1$ of (B).

(B) $\gamma < 1$ (in particular, $\eta_* = 0$), *the bounded case.*
This is the case of islands of rapidly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ at least as fast as some power of t. Here the potential ξ is necessarily bounded from above. This case was treated for a special case in [A95] and in generality in [BK01].

In all these cases, the asymptotics of the moments of $U(t)$ are, for any $p \in (0,\infty)$, given by

$$\frac{1}{pt}\log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\big(\chi + o(1)\big), \qquad \text{as } t\uparrow\infty, \tag{2.25}$$

where χ is alternately given in terms of two characteristic variational principles, which describe the shape of the potential and the local times that give the main contribution to $\langle U(t)^p\rangle$, respectively. Let us give some more insight in the four cases.

Remark 2.10. (The case (SP).) This case is included in [GM98] as the upper boundary case $\rho = \infty$ in their notation. Here $\chi = 2d$, which is identical to the value of the right-hand side of (2.27) for $\varrho = \infty$. The scale function $\alpha(t) = 1$ is constant, and the first term on the right hand side in (2.25) dominates the sum, which diverges to infinity. ◇

Remark 2.11. (The case (DE).) The study of this case was initiated in [GM98]. The particular interest of this class comes from the fact that the intermittent islands have a discrete and non-trivial structure, since $\alpha(t)$ stays bounded (and may be put equal to one). The main representative of this class is the *double-exponential distribution* (which is indeed a reflected Gumbel distribution) given by

$$\mathrm{Prob}(\xi(0) > r) = \exp\big\{-\mathrm{e}^{r/\varrho}\big\}, \qquad r\in\mathbb{R}, \tag{2.26}$$

where $\varrho\in(0,\infty)$ is a parameter. The characteristic variational problem is given as

$$\chi = \inf_{g\in\ell^2(\mathbb{Z}^d)\colon \|g\|_2=1}\big[\langle g, -\Delta^{\mathrm{d}} g\rangle + \varrho I(g^2)\big], \qquad \text{where } I(g^2) = -\sum_{z\in\mathbb{Z}^d} g^2(z)\log g^2(z). \tag{2.27}$$

It is known that this formula possesses minimizers, which are unique (up to spatial shifts) for sufficiently large ϱ. These minimizers are not explicitly known, but they are known to approach delta-like functions for $\varrho\uparrow\infty$ and Gaussian functions (after rescaling) for $\varrho\downarrow 0$. This both is consistent with the understanding that (SP) is the boundary case of (DE) for $\varrho\uparrow\infty$, and (AB) is the boundary case of (DE) for $\varrho\downarrow 0$. The dual formula for χ is

$$\chi = \tag{2.28}$$

◇

Remark 2.12. (The case (AB).) One obtains examples of potentials (unbounded from above) by replacing ϱ in (2.26) by a sufficiently regular function $\varrho(r)$ that tends to 0 as $r\to\infty$, and other examples (bounded from above) by replacing γ in (2.31) by a sufficiently regular function $\gamma(x)$ tending to 1 as $x\downarrow 0$. We find that $\widehat{H}(y) = \mathrm{const}\, y\log y$, and the rate function in (2.9) turns out to be

$$I_R(\varphi) = \mathrm{const}\int_{Q_R}\mathrm{e}^{\varphi(x)/\varrho}\,\mathrm{d}x. \tag{2.29}$$

The characteristic variational problem is given by

$$\chi = \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1}\Big[\|\nabla g\|_2^2 + \varrho\int_{\mathbb{R}^d} g^2\log g^2\Big]. \tag{2.30}$$

This is easily seen to be (up to spatial shifts, uniquely) minimised by the Gaussian density $g^2(x) = \mathrm{const}\,\mathrm{e}^{-\varrho\|x\|_2^2}$, which is the principal eigenfunction of $\Delta+\varphi$ for the parabolic function $\varphi(x) = \mathrm{const} - \varrho\|x\|_2^2$. The parabola in turn is the (up to spatial shifts, unique) minimiser of the alternate representation of χ. Hence, in spite of a relatively odd definition of the potential distribution, the appropriately rescaled and shifted shape of the local times and of the potential that give the main contribution to the moments of the total mass are unique, explicit and elementary functions. $\diamond$

Remark 2.13. (The case (B).) The main representatives of this case have (without loss of generality) essential supremum equal to zero and are given by

$$\mathrm{Prob}(\xi(0) > -x) = \exp\big\{-Dx^{-\frac{\gamma}{1-\gamma}}\big\}, \qquad x \in (0,\infty), \tag{2.31}$$

where $D \in (0,\infty)$ and $\gamma \in [0,1)$ are parameters. We find that $H(t) \approx -\mathrm{const}\, t^\gamma$ and that $\widetilde{H}(y) = \frac{\rho}{\gamma-1}\,(y^\gamma - y)$. Then we have

$$\chi = \inf_{g\in H^1(\mathbb{R}^d)\colon \|g\|_2=1} \Big\{\|\nabla g\|_2^2 - \rho\int_{\mathbb{R}^d} \frac{g^{2\gamma}-g^2}{\gamma-1}\Big\}. \tag{2.32}$$

In the lower boundary case where $\gamma = 0$, the functional $\int g^{2\gamma}$ must be replaced by the Lebesgue measure of $\mathrm{supp}\,(g)$. The formula is well-known and well-understood. In particular, the minimizer exists, is unique up to spatial shifts, and has compact support. This is classic for $\gamma = 0$ and was recently worked out in [S11]. The boundary $\gamma \uparrow 1$ connects up smoothly with the case (AB), as can also be seen from a comparison of (2.32) with (2.27). $\diamond$

Remark 2.14. (Spatially continuous counterparts.) Apparently, the distinction depends on just one single random variable $\xi(0)$. In the discrete setting, one naturally covers all i.i.d. potentials, but in the continuous case, one cannot do this so easily without determining the spatial correlations. Nevertheless, a number of potentials studied in the literature obviously belong to one of the above classes in a phenemonological sense. E.g., the case of a Poisson field of obstacles and many variants belong to the case (B) (studied in [S98], [A94] and [A95] and more), the case of a Poisson field with positive cloud and many Gaussian fields belong to (SP) (see [GK00] and [GKM00], e.g.) belong to (SP). $\diamond$

3. Almost sure logarithmic asymptotics for the total mass

In this section, we explain the basic picture that underlies the almost sure asymptotics of the total mass $U(t)$. We give first in Section 3.1 a heuristics based on the eigenvalue expansion introduced in Remark 1.7 and in Section 3.2 a precise formulation in the four respective cases of potentials.

3.1 Heuristic derivation

Here we explain, again on a heuristic level, the almost sure asymptotics of $U(t)$ as $t \to \infty$. Because of (2.2), it suffices to study the asymptotics of the principal eigenvalue $\lambda_t(\xi)$. We suppose that the potential distribution satisfies the same as in Section 2.1. For definiteness, we assume that $\alpha(t) \to \infty$ as $t \to \infty$, but the same heuristics applies in all other cases.

In order to shift the potential down to size of finite order, we subtract its maximum $h_t = \max_{z\in B_t} \xi(z)$ and study $\xi - h_t$. As we explained in Remark 1.8, the main contribution to $\lambda_t(\xi)$ comes from islands whose radius is of a certain deterministic, time-depending order, which we denote $\widetilde{\alpha}(t)$. This scale is determined by a balance between the inverse probability that the principal eigenvalue of $\Delta^{\mathrm{d}} + \xi - h_t$ in a local box of radius $\approx \widetilde{\alpha}$ is of finite order and the number of such microboxes, which is roughly t^d. Note that the inverse probability to have such an eigenvalue in a box of radius $\alpha(t)$, is of exponential order $t/\alpha(t)^2$, as we have seen in Section 2.1. Hence, we have to pick $\widetilde{\alpha}(t)$ as some $\alpha(\beta(t))$ (i.e., substitute t by the scale $\beta(t)$) where $\beta(t)$ is chosen such that

$$\frac{\beta(t)}{\alpha(\beta(t))^2} = d \log t, \tag{3.1}$$

where we approximated the number of microboxes with t^d, and then we pick $\widetilde{\alpha}(t) = \alpha(\beta(t))$. In other words, $t \mapsto \beta(t)$ is the inverse of the map $t \mapsto t/\alpha(t)^2$, evaluated at $d \log t$. With these choices, one can expect that, with high probability, in at least one of the microboxes (actually, in many of them), $\xi - h_t$ is of order one. Note that $\widetilde{\alpha}(t)$ is logarithmic in t and hence much smaller than $\alpha(t)$. As $t \to \infty$, $\widetilde{\alpha}(t)$ tends to zero, one, or ∞, respectively, if $\alpha(t)$ does. It is not too difficult to identify the asymptotics of h_t as

$$h_t = \frac{H(t/\widetilde{\alpha}(t)^d)}{t/\widetilde{\alpha}(t)^d} + o(\widetilde{\alpha}(t)^{-2}), \qquad t \to \infty, \text{almost surely.} \tag{3.2}$$

More precisely, the relevant islands ('microboxes') have radius $R\widetilde{\alpha}(t)$, where R will be chosen large afterwards. Let $z \in B_t$ denote the center of one of these islands $\widetilde{B} = z + B_{R\widetilde{\alpha}(t)}$ meeting the two requirements (1) the potential ξ is of order h_t in $\widetilde{B}$ and (2) $\xi - h_t$ approximates an optimal shape within $\widetilde{B}$, after rescaling. Hence, we consider the shifted and rescaled field in the box $\widetilde{B}$,

$$\overline{\xi}_t(\cdot) = \widetilde{\alpha}(t)^2 \Big[\xi\big(z + \cdot\, \widetilde{\alpha}(t)\big) - h_t\Big], \qquad \text{in } Q_R = (-R, R)^d. \tag{3.3}$$

Note that

$$\overline{\xi}_t \approx \varphi \quad \text{in } Q_R \qquad \Longleftrightarrow \qquad \xi(z + \cdot) \approx h_t + \tfrac{1}{\widetilde{\alpha}(t)^2} \varphi\big(\tfrac{\cdot}{\widetilde{\alpha}(t)}\big) \quad \text{in } \widetilde{B} - z. \tag{3.4}$$

Now pick any shape function $\varphi \colon Q_R \to \mathbb{R}$ such that $I_R(\varphi) < 1$, where I_R is the rate function of the large deviation principle in (2.9). The condition $I_R(\varphi) > 1$ means that the shape φ is not too improbable for the potential distribution. A crucial Borel-Cantelli argument shows that, for a given shape φ, with probability one, for any t sufficiently large, there exists at least one $z \in B_t$ such that in the box $\widetilde{B} = z + B_{R\widetilde{\alpha}(t)}$ the event $\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}$ occurs.

Hence, with probability one, for all large t, there is at least one box $\widetilde{B}$ in which the potential looks like the function on the right of (3.4). The contribution to $\lambda_t(\xi)$ coming from one of the boxes $\widetilde{B}$ is equal to the associated principal eigenvalue

$$\lambda_{\widetilde{B}-z}\big(h_t + \tfrac{1}{\widetilde{\alpha}(t)^2}\varphi\big(\tfrac{\cdot}{\widetilde{\alpha}(t)}\big)\big) \approx h_t + \frac{1}{\widetilde{\alpha}(t)^2} \lambda_R^{(\mathrm{c})}(\varphi), \tag{3.5}$$

where we recall that $\lambda_R^{(\mathrm{c})}(\varphi)$ is the principal Dirichlet eigenvalue of the operator $\kappa\Delta + \varphi$ in the cube Q_R. Onthe event $\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}$, obviously, $\lambda_t(\xi)$ is asymptotically not smaller than the expression on the right of (3.5). Starting from the Feynman-Kac formula in (1.5), we obatin a lower estimate by inserting the indicator on the event that the random path moves quickly to the box $\widetilde{B}$ (this is done at a negligible cost) and stays all the time until t in that box (the cost for doing this is $\exp\{-t\widetilde{\alpha}(t)^{-2}\lambda_R^{(\mathrm{c})}(\varphi)\}$ to high precision). Summarizing, we have argued that

$$\frac{1}{t} \log U(t) \approx \lambda_t(\xi) \geq \frac{H(t/\widetilde{\alpha}(t)^d)}{t/\widetilde{\alpha}(t)^d} - \frac{1}{\widetilde{\alpha}(t)^2} \widetilde{\chi}, \qquad t \to \infty, \tag{3.6}$$

where $\widetilde{\chi}$ is given in terms of the characteristic variational problem

$$\widetilde{\chi} = \lim_{R \to \infty} \inf_{\varphi \colon Q_R \to \mathbb{R}, I_R(\varphi) < 1} \left[-\lambda_R^{(\mathrm{c})}(\varphi)\right]. \tag{3.7}$$

This ends the heuristic derivation of an almost sure lower bound for $U(t)$, which is sharp, see Remark 3.1 below. Like in the annealed case, there are two terms, which describe the absolute height of the potential in the 'macrobox' B_t, and the shape of the potential in the relevant 'microbox' $\widetilde{B}$, more precisely the spectral properties of $\kappa\Delta + \xi$ in that microbox. The interpretation is that, for R large and φ_* an approximate minimizer in (3.7), the main contribution to $U(t)$ comes from a small box $\widetilde{B}$ in B_t, with radius $R\widetilde{\alpha}(t)$, in which the shifted and rescaled potential $\overline{\xi}_t$ looks like φ_*. The condition $I_R(\varphi_*) < 1$ guarantees the existence of such a box, and $\lambda_R(\varphi_*)$ quantifies the contribution from that box.

Remark 3.1. (The upper bound in (3.6)**.)** Like for the moment asymptotics, the proof of good upper bounds is technically more involved and more abstract, since one has to take care of *all* the paths, not only some optimal ones. Some of the methods outlined in Remarks 2.3 and (2.6) are helpful also for the proof of the upper bound in (3.6). Indeed, e.g., the upper estimate of the principal eigenvalue in a macrobox against the maximum of local principal eigenvalues in mutually overlapping microboxes (developed in [GK00] and [BK01]) reduces the proof to the control of the maximum of roughly t^d independent copies of $\lambda_{R\widetilde{\alpha}(t)}(\xi)$, which is easy to handle. Also variants of Sznitman's method of enlargement of obstacles [S98], [A94] are suitable to yield a proof in the cases of Brownian motion among Poisson obstacles and survival problems for the simple random walk, respectively. Note however that the periodisation method introduced in [DV75] is not suitable, since it would map *all* the microboxes onto one place. $\diamond$

Remark 3.2. (Relation between the variational formulas χ and $\widetilde{\chi}$.) The variational formulas in (3.7) and (2.12) are in close connection to each other. In particular, it can be shown on a case-by-case basis that the minimizers of (3.7) are rescaled versions of the minimizers of (2.12) in the cases (B) and (AB), and they are even identical in the cases (SP) and (DE). This means that, up to rescaling, the optimal potential shapes in the annealed and in the quenched setting are identical. $\diamond$

3.2 The result and comments

Here we give more details and comments on the almost sure asymptotics in the four casesof potentials introduced in Section 2.2.

Remark 3.3. (The case(DE).) Here $0 \leq \widetilde{\chi}_d \leq 2d\kappa$. An analytic description of $\widetilde{\chi}_d$ is as follows. Define $I\colon [-\infty, 0]^{\mathbb{Z}^d} \to [0, \infty]$ by

$$I(V) = \begin{cases} \sum_{x\in\mathbb{Z}^d} e^{V(x)/\varrho}, & \text{if } \varrho \in (0,\infty), \\ |\{x \in \mathbb{Z}^d\colon V(x) > -\infty\}|, & \text{if } \varrho = \infty. \end{cases} \tag{3.8}$$

One should regard I as *large deviation rate function* for the fields $\xi - h_t$ (recall (2.9) and note that $\alpha(t) = 1$ here). Indeed, if the distribution of ξ is exactly given by (2.26), then we have

$$\text{Prob}\left(\xi(\cdot) - h > V(\cdot) \text{ in } \mathbb{Z}^d\right) = \exp\left\{-e^{h/\varrho} I(V)\right\}$$

for any $V\colon \mathbb{Z}^d \to [-\infty, 0]$ and any $h \in (0,\infty)$. For $V \in [-\infty, 0]^{\mathbb{Z}^d}$, let $\lambda(V) \in [-\infty, 0]$ be the top of the spectrum of the self-adjoint operator $\kappa\Delta + V$ in the domain $\{V > -\infty\}$ with zero boundary condition. In terms of the Rayleigh-Ritz formula,

$$\lambda(V) = \sup_{f\in\ell^2(\mathbb{Z}^d)\colon \|f\|_2=1} \big\langle(\kappa\Delta + V)f, f\big\rangle, \tag{3.9}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ denote the inner product and the norm in $\ell^2(\mathbb{Z}^d)$, respectively. Then

$$-\widetilde{\chi}_d = \sup\Big\{\lambda(V)\colon V\in[-\infty,0]^{\mathbb{Z}^d},\, I(V)\le 1\Big\}. \tag{3.10}$$

This variational problem is 'dual' to the variational problem (2.27) and, in particular, $\widetilde{\chi}_d=\chi_d$. ◇

4. Mass concentration

One of the basic questions is about where the main heat flow is located at large times, see question (2) in Section 1.1. One is interested to determine some (random and time-dependent) regions that carry the total mass up to some negligible amount. This question is interesting in both the annealed and the quenched setting. From the heuristics in Section 3.1 and also from the proofs in the original literature, we already see that, almost surely, the mass coming from the union of the intermittent islands (see Remark 1.8) is asymptotically equal to the total mass, but this result is weak in view of the logarithmic error and because there is no assertion about the negligibility of the contribution coming from the complement of the union of the intermittent islands. The goal here is rather to find relatively small and few random subsets $B_1,\dots,B_{n_t}$ of $\mathbb{Z}^d$ such that, almost surely,

$$U(t)\sim\sum_{i=1}^{n_t}\sum_{z\in B_i}u(t,x), \qquad t\to\infty, \tag{4.1}$$

in the sense of asymptotic equivalence. This is a strong assertion of *mass concentration* for the PAM. Certainly, the sets B_i should be equal to the set on which the leading eigenfunctions of the Anderson operator $\mathcal{H}$ are concentrated, see Remark 1.8.

Results of this type have been derived in the literature in the cases of Brownian motion among Poisson obstacles [S98], see Remark 4.2, for the double-exponential distribution [GKM07] and for the (much less demanding) Pareto distribution [KLMS09], in which an extremely strong concentration can be proved: it turns out that n_t can be picked as $n_t=2$, and the sets B_1 and B_2 are singletons. We will explain the case of the double-exponential distribution in Section 4.1 in greater detail and the case of the Pareto distribution in Section 4.2.

However, the concentration property in *just one island* is conjectured to hold in much greater generality, but there is no proof of such an assertion in general in the literature yet. However, in Section 4.3 we will explain this effect and elements of the proof for the double-exponential distribution, which is due to future work [BK11+].

4.1 Geometric characterisation of intermittency

In this section we give a more precise formulation of (4.1) for the case of the double-exponential distribution introduced in Remark 2.11. In particular, we will describe the sets B_i and the typical shape of the potential ξ and of the solution $u(t,\cdot)$ inside these sets, almost surely. We follow [GKM07], but slightly simplify some facts.

For simplification, we also assume that the parameter ϱ appearing in (2.26) is so large that, up to spatial shifts, the variational problem in (2.27) possesses a unique maximizer, which has a unique maximum [GH99]. By V_* we denote the unique maximizer of (3.10) which attains its unique maximum at the origin. We will call V_* *optimal potential shape.* Some crucial properties of the formula (3.10) are as follows. The operator $\Delta^{\rm d}+V_*$ has a unique nonnegative eigenfunction $w_*\in\ell^2(\mathbb{Z}^d)$ with $w_*(0)=1$ corresponding to the eigenvalue $\lambda(V_*)$. Moreover, $w_*\in\ell^1(\mathbb{Z}^d)$ is positive everywhere on $\mathbb{Z}^d$.

One crucial object is the maximum $h_t = \max_{z\in B_t} \xi(z)$ of the potential in the macrobox B_t. We shall see that the main contribution to the total mass $U(t)$ comes from a neighbourhood of the set of best local coincidences of $\xi - h_t$ with spatial shifts of V_*. These neighborhoods are widely separated from each other and hence not numerous. We may restrict ourselves further to those neighborhoods in which, in addition, $u(t,\cdot)$, properly normalized, is close to w_*.

Denote by $B_R(y) = y + B_R$ the closed box of radius R centered at $y \in \mathbb{Z}^d$ and write $B_R(A) = \bigcup_{y\in A} B_R(y)$ for the R-box neighborhood of a set $A \subset \mathbb{Z}^d$. In particular, $B_0(A) = A$.

For any $\varepsilon > 0$, let $r(\varepsilon, \varrho)$ denote the smallest $r \in \mathbb{N}_0$ such that

$$\|w_*\|_2^2 \sum_{x\in\mathbb{Z}^d\setminus B_r} w_*(x) < \varepsilon. \tag{4.2}$$

Given $f\colon \mathbb{Z}^d \to \mathbb{R}$ and $R > 0$, let $\|f\|_R = \sup_{x\in B_R} |f(x)|$. The main result of [GKM07] is the following.

Theorem 4.1. *There exists a random time-dependent subset $\Gamma^* = \Gamma^*_{t\log^2 t}$ of $B_{t\log^2 t}$ such that, almost surely,*

$$(i) \qquad \liminf_{t\to\infty} \frac{1}{U(t)} \sum_{x\in B_{r(\varepsilon,\varrho)}(\Gamma^*)} u(t,x) \geq 1-\varepsilon, \qquad \varepsilon\in(0,1); \tag{4.3}$$

$$(ii) \qquad |\Gamma^*| \leq t^{o(1)} \quad \text{and} \quad \min_{y,\widetilde{y}\in\Gamma^*\colon y\neq\widetilde{y}} |y-\widetilde{y}| \geq t^{1-o(1)} \qquad \text{as } t\to\infty; \tag{4.4}$$

$$(iii) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \big\|\xi(y+\cdot) - h_t - V_*(\cdot)\big\|_R = 0, \qquad R>0; \tag{4.5}$$

$$(iv) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \Big\|\frac{u(t,y+\cdot)}{u(t,y)} - w_*(\cdot)\Big\|_R = 0, \qquad R>0. \tag{4.6}$$

Theorem 4.1 states that, up to an arbitrarily small relative error ε, the islands with centers in Γ^* and radius $r(\varepsilon,\varrho)$ carry the whole mass of the solution $u(t,\cdot)$. (In other words, in terms of (4.1), $n_t = |\Gamma^*| = t^{o(1)}$, and the B_i are the R-neighbourhoods of the sites in Γ^*.) Locally, in an arbitrarily fixed R-neighborhood of each of these centers, the shapes of the potential and the normalized solution resemble $h_t + V_*$ and w_*, respectively. The number of these islands increases at most as an arbitrarily small power of t and their distance increases almost like t.

The main strategy of [GKM07] is not based on the eigenvalue expansion in Remark 1.7, since it is difficult to handle the possible negativity of the eigenfunctions at zero. Instead, a strategy is developed that works exclusively with *principal* eigenfunctions of $\Delta^{\rm d} + \xi$ in local neighbourhoods of high exceedances of the potential, after destroying the quality of the eigenvalues in all the other islands. One crucial point is the proof of the exponential localisation of the corresponding eigenfunctions using a decomposition technique for the paths in the Feynman-Kac representation for these principal eigenfunctions (called *probabilistic cluster expansion* in [**?**]). There is no control on the differences between any two of the top eigenvalues, but it is shown that the concentration centres of these eigenfunctions have mutual distance $t^{1-o(1)}$ from each other. This in turn implies that there are not more than $t^{o(1)}$ of them, and therefore there must be, somewhere close to the top of the spectrum, some gap of minimal size $t^{-o(1)}$. This gap played a crucial rôle in the proof of the exponential localisation.

An optimisation on n_t was left open in [GKM07], but in Section 4.3 we will see that, under some additional regularity assumption, it should be possible to pick n_t as just one in a certain sense.

Remark 4.2. (Brownian path concentration among Poisson obstacles.) In the case of Brownian motion among Poisson obstacles, see Remark 1.5, an assertion was proved that is closely related to

Theorem 4.1. Sznitman was interested in the behaviour of the motion rather than in the mass, hence, he did not prove a mass concentration property. His main result here can be formulated as follows. Almost surely, as $t \to \infty$, there are $n_t = t^{o(1)}$ balls $B_1, \dots, B_{n_t} \subset \mathbb{R}^d$ of radius const $\widetilde{\alpha}(t)$ with mutual distance $t^{1-o(1)}$ such that the Brownian motion of the Feynman-Kac formula does the following with probability (under the transformed path measure) tending to one. It arrives, after some deterministic diverging time $\ll t$ at one of these balls and does not leave it up to time t. These balls are the regions in which the local Dirichlet eigenvalues of $\Delta + \xi$ attains its largest values inside a macrobox $B_{t \log^2 t}$.$\diamondsuit$

Both results in [GKM07] and [S98] do not take into account the distance to the origin of the localisation centres of the leading eigenfunctions, but optimise only over the local eigenvalues, provided their value is not too far from the maximum. This may be the reason that the number n_t of relevant island can be bounded by $t^{o(1)}$ only. A finer argument will be detailed in Section 4.3.

4.2 Potentials without exponential moments: a 'two-cities theorem'

The structure of the intermittent islands is the simplest for the class of single-peak potentials, where the single-site potentail distribution possesses the thickest tails, see Remark 2.10. Here the islands are just single sites. This class can easily be extended to the class of potentials having so thick tails that their positive exponential moments are all infinite. Examples comprise the *Weibull tails*, where $\text{Prob}(\xi(0) > r) \approx \exp\{-r^\gamma\}$ with $\gamma \in (0,1)$, and the *Pareto distribution*, where $\text{Prob}(\xi(0) > r) = r^{-\alpha}$ for $r \in [1, \infty$ with a parameter $\alpha > d$. Since these distributions have no finite exponential moments, the moments of the total mass $U(t)$ are infinite for any $t > 0$, and we cannot speak about moment asymptotics.

However, it turns out that an intermittency picture is present and is even more pronounced and can be proved with less technicalities than in the case (SP). Indeed, a very strong form of the concentration property in (4.1) can be proved: For Pareto-distributed potentials, (4.1) is true with $n_t = 1$ and a singleton B_1, if the limit is understood in probability, and with $n_t = 2$ and two singletons B_1 and B_2 almost surely. This is remarkable, since it is the strongest assertion possible on the long-time behaviour of the PAM, and it opens up the possibility to analyse properties of the entire heat flow that is described by the process $(u(t,\cdot))_{t\in(0,\infty)}$ just in terms of the concentration site as a stochastic process in t. In particular, ageing properties can be studied, see Section 5.1 below.

Let us briefly summarize what has been achieved for the PAM with thick-tailed potentials. Since there is a recent survey [M11] on this special aspect of the PAM, we abstain from an extensive formulation.

The study of the PAM with thick-tailed potentials was initiated in [HMS08], where almost sure and distributional limit theorems for the total mass $U(t)$ are derived for the Weibull and the Pareto case. We discuss here only the Pareto distribution, since this the only of these cases whose analysis has been pushed through to an extremely high level (even though similar assertions are believed to be true for the Weibull distribution ,e.g.) For the Pareto distribution, it is proved that

$$\frac{(\log t)^{\frac{d}{\alpha-d}}}{t^{\frac{\alpha}{\alpha-d}}} \log U(t) \Longrightarrow Y, \qquad \text{where } \mathbb{P}(Y \le y) = \exp\{-\theta y^{d-\alpha}\}, \tag{4.7}$$

and θ is some explicit constant, and some almost sure liminf and limsup results for the logarithm of $\frac{1}{t} \log U(t)$ are derived. Note that the limiting distribution in (4.7) is the Fréchet distribution, one of the three famous limiting distribtuions for the maximum of i.i.d. random variables. Hence, the assertion of (4.7) is very much in line with the understanding that all the leading eigenfunctions in the expansion (1.8) are delta-like functions with extremly high values, and therefore $U(t)$ is approximately equal to the maximum of a large number of i.i.d. Pareto-distributed random variables. Since it is known that the difference between the largest and the second-largest of such random variables is huge, one can hope that this huge difference can be used to show that the main contribution ot $U(t)$ comes just from *one* of these values, i.e., from one site.

In the follow-up paper [KLMS09], techniques from [GKM07] were added to prove that the above scenario can indeed be proved. More precisely, it was proved that there is a stochastic process $(Z_t)_{t\in(0,\infty)}$ in $\mathbb{Z}^d$ such that

$$U(t) \sim u(t, Z_t) \qquad \text{as } t \to \infty \text{ in probability.} \tag{4.8}$$

This is the announced strong form of (4.1). An informal description of the site Z_t is as follows. Consider the function

$$\Psi_t(z) = \xi(z) - \frac{|z|}{t} \log \frac{|z|}{2det}, \qquad z \in \mathbb{Z}^d, t > 0,$$

then $e^{t\Psi_t(z)}$ is roughly equal to the contribution to the Feynman-Kac formula in (1.5) coming from a path that quickly runs to the site z and stays in z for the rest of the time until t. (The first term is the potential value that is attained for $\approx t$ time units, and the second is the probability to go for a distance $|z|$ in $\approx o(t)$ time units.) Then Z_t is defined as the site that maximises Ψ_t. In particular, $\Psi_t(Z_t) = \max_{z\in\mathbb{Z}^d} \Psi_t(z) \approx \frac{1}{t} \log U(t)$. This description lies at the heart of the heuristics in Remark 1.8 and improves the idea outlined in Section 4.1.

Remark 4.3. (Almost sure concentration.) The asymptotics in (4.8) cannot be true almost surely. In this case, t would be a random time and would also sooner or later attain a value that lies in a time interval during which the dominant potential peak wanders from one location to another one. Such phases of wandering of the overwhelming mass from one 'city' to the next one occur, since the horizon increases as t increases, and the maximisation of the field takes place over larger and larger areas. However, in [KLMS09] it is proved that the main mass is concentrated in no more than *two* sites at any large time t, almost surely. This interpretation gave this section and the paper [KLMS09] their titles. ◇

The proof of (4.8) relies on spectral theory and on techniques from the theory of order statistics for i.i.d. random variables and implicitly on the theory of Poisson point convergence, which was later detailed and further exploited in [MOS11]. We will explain this mechanism in Section 4.3 in greater generality and give here only the main results of [MOS11] and some comments.

The description of the entire process $(Z_t)_{t\in(0,\infty)}$ is identified as follows. In [MOS11] it is proved that there is a (time-inhomogeneous) Markov process $(Y_t^{(1)}, Y_t^{(2)})_{t\in(0,\infty)}$ in $\mathbb{Z}^d \times \mathbb{Z}$ such that, as $T \to \infty$,

$$\Big(\Big(\frac{\log T}{T}\Big)^{\frac{\alpha}{\alpha-d}} Z_{tT}, \Big(\frac{\log T}{T}\Big)^{\frac{d}{\alpha-d}} \xi(Z_{tT}) \Big)_{t\in(0,\infty)} \Longrightarrow \Big(Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha-d} |Y_t^{(1)}| \Big)_{t\in(0,\infty)}.$$

In particular, $(\frac{\log T}{T})^{\frac{\alpha}{\alpha-d}} Z_T$ converges in distribution to $Y = Y_1^{(1)}$, which shows that the macroboxes B_t whose radius were picked of order $\approx t$ for potentials with finite exponential moments in Section 3.1, here have a radius of order $\gg (\frac{T}{\log T})^{\frac{\alpha}{\alpha-d}}$, which is much larger.

Remark 4.4. (Exponential and Weibull distribution.) Another interesting potential distribution that turns out to phenemonologically lie in the class (SP), is the *exponential distribution*, $\mathrm{Prob}(\xi(0) > r) = e^{-r}$ for $r \in (0,\infty)$. This distribution is considered in [LM11], and it is found that a concentration property in one single site takes place as well. First, like for the Pareto distribution in [HMS08], some distributional and almost sure liminf and limsup results for $\frac{1}{t}\log U(t)$ are given in [LM11]. Furthermore, it is shown that the point process

$$\frac{1}{U(t)}\sum_{z\in\mathbb{Z}^d} u(t,z)\delta_{z/r_t}\qquad \text{with } r_t=\frac{t}{\log\log t},$$

converges towards δ_Y, where Y is an $\mathbb{R}^d$-valued random variable with i.i.d. coordinates with exponential distribution with uniform random sign. (According to [LM11], the analogous assertion for the Weibull distribution can be formulated and proved in the same way, the details are not given.) ◇

4.3 Concentration in one island: heuristics

As we argued in Remark 1.8, the main contribution to the total mass $U(t)$ should come from just one island of eigenfunction concentration that optimises the relation between the high value of the corresponding eigenvalue and the proximity of the region to the initial site, the origin. For the class (SP) of thick-tailed potentials, this idea has been turned into rigorous proofs, as we explained in Section 4.2. However, this is a quite simple case, as the islands are just singletons, the eigenfunctions are strongly delta-like, and the differences between the largest and second-largest potential value is huge. The classes (DE), (AB) and (B) are much more interesting, since the intermittent islands carry some non-trivial structure that is asymptotically given by deterministic variational formulas. In this section, we explain the concentration phenomenon in just one island in the interesting case (DE). Furthermore, we outline a proof for this fact, which is inspired by ideas from the theory of Anderson localisation. We follow [BK11+].

The main idea is to achieve some control on the differences between subsequent eigenvalues close to the top of the spectrum of $\mathcal{H}=\Delta^{\rm d}+\xi$ in terms of an *order statistics* for the sequence of eigenvalues in large boxes. Furthermore, one must show that, up to some small error, any global eigenvalue/eigenfunction pair is equal to a principal pair in a local microbox with Dirichlet boundary condition. Then the eigenvalues are practically independent, and their largest eigenvalue should satisfy an order statistics, provided the distribution of the principal eigenvalue lies in the max-domain of attraction of one of the three famous max-distributions. For having this, one needs an assertion of the form

$$\mathrm{Prob}(\lambda_1(B_L)\geq a_L)=\frac{1}{L^d}\qquad\Longrightarrow\qquad \mathrm{Prob}(\lambda_1(B_L)\geq a_L+sb_L)=\frac{\mathrm{e}^{-s}(1+o(1))}{L^d}\ \text{for any } s\in\mathbb{R},$$

as $L\to\infty$, for some scale functions a_L and b_L, where B_L is the centred box in $\mathbb{Z}^d$ with cardinality $\approx L^d$, and $\lambda_k(B)$ the k-th largest eigenvalue of $\mathcal{H}$ in $B\subset\mathbb{Z}^d$.

For the potential given precisely by (2.26), this implication is shown to be true with the values $a_L=\rho\log\log L^d-\chi+o(1)$ and $b_L=1/\log L$. This is the core of a proof not only of an eigenvalue order statistics in the domain of attraction of the Gumbel distribution, but even of the convergence of the point process of rescaled eigenvalues, together with the rescaled localisation centres of the corresponding eigenfunctions, towards an explicit Poisson point process. A (slightly imprecise) formulation is the following. Let $v_B^{(k)}$ denote an ℓ^2-normalised eigenfunction of $\mathcal{H}$ corresponding to $\lambda_k(B)$.

Theorem 4.5 (Eigenvalue order-statistics). *For each $L\geq 1$ there is a sequence $X_1,X_2,\dots$ of random sites in B_L and a number $a_L=\rho\log\log L^d-\chi+o(1)$ such that, for any $R_L\to\infty$,*

$$\sum_{z\colon |z-X_k|\leq R_L}\big|v_{B_L}^{(k)}(z)\big|^2 \underset{L\to\infty}{\longrightarrow} 1 \tag{4.9}$$

in probability, for each $k \in \mathbb{N}$. Moreover, the law of

$$\Big\{\frac{X_k}{L}, \big(\lambda_k(B_L) - \rho \log\log L^d + \chi\big) \log L\Big\}_{k\in\mathbb{N}} \tag{4.10}$$

converges weakly to the ranking, by the value of the last coordinate, of a Poisson point process on $D \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

In particular, any two neighboring eigenvalues have distance of order $1/\log L$ to each other, $(\lambda_1(B_L) - \rho \log\log L^d + \chi) \log L$ converges weakly towards a standard Gumbel distribution, and the eigenfunction localisation centres converge towards a standard Poisson process.

Remark 4.6. (Which max-domain?.) Note from Section 4.2 that the local eigenvalues turned out, in the case(SP), to lie in the max-domain of a Fréchet distribution (a one-sided max-domain that appears if the distribution of maxima is close to the boundary of the support), while for the case (DE) it is the Gumbel distribution, which arises if the maxima have a certain distance to the boundary. On base of the work in [BK11+], we conjecture that a similar picture can be proved for the distributions in the classes (B) and (AB), at least for some prominent representatives. In particular, we conjecture that in both these cases the eigenvalues lie in the max-domain of a Gumbel distribution. However, this should hold in the case (B) only for $\gamma > 0$, where the potential distribution does not feel the boundary of its support immediately. The case $\gamma = 0$ should lead to the Weibull distribution ◇

One of the main technical points in the proof of Theorem 4.5 is the proof of eigenfunction localisation if the eigenvalue is large enough and sufficiently far from all the other large eigenvalues. Here an argument is employed that shows that the eigenfunction remains practically unchanged if the potential is continuously shifted to $-\infty$ outside of a neighbourhood of the local island of sites that give extremely high potential values and carries some mass $\geq 1/2$ of the eigenfunction.

Remark 4.7. (Anderson localisation via Poisson point processes.) It is instructive to compare (4.10) with a related result from the theory of Anderson localisation from [GK11]. That result works in the bulk of the spectrum of $\mathcal{H}$ (rather than at the top), assumes that the potential distribution has a bounded density and that Anderson localisation holds in that spectral interval, picks a growing number of eigenvalues and corresponding eigenfunction centres and shows that their point process, after rescaling, converges towards a standard Poisson process with intensity measure $\mathrm{d}x \otimes \mathrm{d}\lambda$. This convergence does *not* come from a maximisation, and the limiting eigenvalue distribution is not a max-stable distribution, but just the exponential one. Furthermore, the difference between any two subsequent eigenvalues is of order $1/|B_L|$ rather than $1/\log L$. ◇

Having now control on the top eigenvalues and their eigenfunctions, we can turn to the Cauchy problem in (1.1) and couple the eigenvalue order statistics with time. It turns out that L has to be picked of order $t/\log t \log\log\log t$ in order to match the scales. Indeed, put

$$\Psi_{L,t}(\lambda, r) = \frac{t}{r_L}(\lambda - a_L)\frac{1}{b_L} - \frac{|z|}{L}, \qquad \text{where } r_L = L \log L \log\log\log L,$$

and pick $k \in \mathbb{N}$ as the maximiser of $k \mapsto \Psi_{L,t}(\lambda_k(B_L), X_k)$ and put $Z_{L,t} = X_k/L$. As we explained above for the Pareto distribution, the two terms of $\Psi_{L,t}$ describe the exponential rate for the gain of a large eigenvalue and the distance to the origin, respectively. Then, with $L = L_t = t/\log t \log\log\log t$,

$$\lim_{t\to\infty} \frac{1}{U(t)} \sum_{z\colon |z - LZ_{L_t,t}| \leq R_t} u(t,z) = 1 \qquad \text{in probability,} \tag{4.11}$$

for any $R_t \gg \log t$. Furthermore, as $L \to \infty$, the process $(Z_{L,sr_L})_{s\in[0,\infty)}$ converges towards the process of maximisers of the map $z \mapsto t\lambda - |z|$ over a Poisson process on $B_1 \times (0,\infty)$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$. (4.11) specifies the concentration property of (4.1) for the double-exponential distribution.

The proof of (4.11) needs quite some technical work, since the gap between the centred box with radius of order $t/\log t \log\log\log t$ (in which the eigenvalue order statistics holds and therefore the control on their gaps and the eigenfunction localisation is perfect) and the outside of the box with radius of order $t\log^2 t$ (where rough arguments suffice to show that this region is negligible) must be closed. Additional work must be done (and the assertion must be slightly adapted) if the convergence in probability should be strenghened to almost sure convergence.

5. Relations to other models, and future directions

5.1 Time-correlations and ageing

Summary of [GS11a] and [MOS11]

5.2 Acceleration and deceleration

Summary of [S10] respectively [KS11]

5.3 ??? Directed Polymers in random environment

5.4 Transition between quenched and annealed behaviour

drei Artikel von Ben Arous, Ramirez und Molchanov ([BAMR05],[BAMR07]) (DRITTEN HABE ICH NOCH NICHT GEFUNDEN) sowie [GS11b]

5.5 Time-dependent potential

We obtain an often considered extension to the problem (1.1) by allowing for time-dependent potentials, i.e., considering the solution to the cauchy problem

$$\frac{\partial}{\partial t}u(t,z) = \Delta^{\mathrm{d}}u(t,z) + \xi(t,z)u(t,z), \qquad \text{for } (t,z) \in (0,\infty)\times\mathbb{Z}^d, \tag{5.1}$$

$$v(0,z) = v_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{5.2}$$

where $\xi : [0,\infty]\times\mathbb{Z}^d \to \mathbb{R}$ and v_0 is the initial datum. In this case, the corresponding Feynman-Kac representation reads

$$u(t,x) = \mathbb{E}_x\Big[\exp\Big\{\int_0^t \xi(t-s,X(s))\,\mathrm{d}s\Big\}\delta_0(X(t))\Big], \qquad (t,x)\in[0,\infty)\times\mathbb{Z}^d, \tag{5.3}$$

compare e.g. [GdH06]. An important class of time-dependent potentials includes those modeling one or many randomly moving particles acting as catalysts or traps, i.e.,

$$\xi(t,z) = \gamma\sum_{i\in I}\delta_z(Y_t^i)$$

where $\gamma \in \mathbb{R}$ is the so-called coupling constant, I is a possibly random index set and the Y^i are independent simple symmetric random walks starting from possibly random points. These random walks are interpreted as the positions of catalyst particles (the case $\gamma > 0$) or moving traps (if $\gamma < 0$). The simplest case in this class of potentials is the case of a single randomly moving catalyst or trap. Annealed asymptotics of the total mass and intermittency in this model under homogeneous initial conditions (i.e., $v_0 \equiv 1$) have been analysed in [GH06] and [SW11], quenched asymptotics have been adressed in [**?**]. The case of infinitely many randomly moving catalysts starting from a Poisson random field has been considered in [GdH06]. Here, beyond intermittency, the authors also consider an effect called *catalyticity.*

In [GdH06], [GdHM11] and [KS03], for instance, the authors investigate the case of infinitely many randomly moving catalysts. In [CGM11] the authors deal with the case of finitely many catalysts, whereas the article [DGRS11] is dedicated to a model similar to the case of infinitely many moving traps. Further examples of time dependent potentials can be found in [GdHM07], [GdHM09a], [GdHM10], [MMS11], and the recent survey [GdHM09b]. Within these proceedings, [KS11], [LM11] and [MZ11] deal with the parabolic Anderson model with time-independent potential.

Bibliography

[A58] P.W. Anderson, Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505 (1958).

[A94] P. Antal, *Trapping Problems for the Simple Random Walk,* Dissertation ETH Zürich, No. 10759 (1994).

[A95] P. Antal, Enlargement of obstacles for the simple random walk. *Ann. Probab.* **23:3**, 1061–1101 (1995).

[BAMR05] G. Ben Arous, S. Molchanov and A. Ramirez, Tansition from the annealed to the quenched asymptotics for a random walk on random obstacles, *Ann. Probab.* **33**, 2149–2187 (2005).

[BAMR07] G. Ben Arous, S. Molchanov and A. Ramirez, Transition asymptotics for reaction-diffusion in random media, In Probability and Mathematical Physics: A Volume in Honor of Stanislav Molchanov, AMS/CRM, 42, 1–40 (2007).

[BGT87] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation.* Cambridge University Press (1987).

[BK01] M. Biskup and W. König, Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29:2**, 636-682 (2001).

[BK11+] M. Biskup and W. König, Eigenvalue order statistics and mass concentration in the parabolic Anderson model. in preparation (2011).

[B94] E. Bolthausen, Localization of a two-dimensional random walk with an attractive path interaction, *Ann. Probab.* **22**, 875–918 (1994).

[BHK07] D. Brydges, R. van der Hofstad and W. König, Joint density for the local times of continuous time Markov chains. *Ann. Probab.* **35:4**, (2007).

[CM94] R. Carmona and S.A. Molchanov, Parabolic Anderson problem and intermittency. Mem. Amer. Math. Soc. **108** no. 518 (1994).

[DZ98] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2^{nd} Edition. Springer, New York (1998).

[DV75] M. Donsker and S.R.S. Varadhan, Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28,** 525–565, 1975.

[DV79] M. Donsker and S.R.S. Varadhan, On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math.* **32**, 721-747 (1979).

[F08] R. Fukushima, Asymptotics for the Wiener sausage among Poissonian obstacles, *Jour. Stat. Phys.* **133:4**, 639-657 (2008).

[F09a] R. Fukushima, Brownian survival and Lifshitz tail in perturbed lattice disorder, *Jour. Func. Ana.* **256:9**, 2867-2893 (2009).

[F09b] R. Fukushima, From the Lifshitz tail to the quenched survival asymptotics in the trapping problem, *Elec. Comm. Prob.* **14**, paper 42, 435-446 (2009).

[FU10] R. Fukushima and N. Ueki, Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice, *Ann. Inst. Henri Poincaré, Probab. Stat.* **11:6**, 1053-1083 (2010).

[FU11] R. Fukushima and N. Ueki, Moment asymptotics for the parabolic Anderson problem with a perturbed lattice potential, *Jour. Func. Anal.* **260:3**, 724-744 (2011).

[G77] J. Gärtner, On large deviations from the invariant measure, *Th. Prob. Appl.* **22**, 24–39 (1977).

[GH99] J. Gärtner and F. den Hollander, Correlation structure of intermittency in the parabolic Anderson model. *Probab. Theory Relat. Fields* **114**, 1–54 (1999).

[GK00] J. Gärtner and W. König, Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10:3**, 192–217 (2000).

[GK05] J. Gärtner and W. König, The parabolic Anderson model. in: J.-D. Deuschel and A. Greven (Eds.), *Interacting Stochastic Systems*, pp. 153–179, Springer (2005).

[GKM00] J. Gärtner, W. König and S. Molchanov, Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Relat. Fields* **118:4**, 547-573 (2000).

[GKM07] J. Gärtner, W. König and S. Molchanov, Geometric characterization of intermittency in the parabolic Anderson model. *Ann. Probab.* **35:2**, 439-499 (2007).

[GM90] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model I. Intermittency and related topics. *Commun. Math. Phys.* **132**, 613–655 (1990).

[GM98] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model II. Second-order asymptotics and structure of high peaks. *Probab. Theory Relat. Fields* **111**, 17–55 (1998).

[GS11a] J. Gärtner and A. Schnitzler, Time correlations for the parabolic Anderson model, preprint (2011).

[GS11b] J. Gärtner and A. Schnitzler, Stable limit laws for the parabolic Anderson model between quenched and annealed behaviour, preprint (2011).

[GK11] F. Germinet and F. Klopp, Spectral statistics for random Schrödinger operators in the localized regime, preprint (2011).

[GK09] G. Grüninger and W. König, Potential confinement property of the parabolic Anderson model, *Ann. Inst. Henri Poincaré, Probab. Stat.* **45:3**, 840–863 (2009).

[HKM06] R. van der Hofstad, W. König and P. Mörters, The universality classes in the parabolic Anderson model. *Commun. Math. Phys.* **267:2**, 307-353 (2006).

[HMS08] R. van der Hofstad, P. Mörters and N. Sidorova, Weak and almost sure limits for the parabolic Anderson model with heavy-tailed potential, *Ann. Appl. Prob.* **18**, 2450-2494 (2008).

[HW94] F. den Hollander and G.H. Weiss, Aspects of trapping in transport processes, *in:* Contemporary problems in Statistical Physics, G.H. Weiss, ed., SIAM, Philadelphia, 1994.

[K08] W. Kirsch, An invitation to random Schrödinger operators, *Panoramas et Syntheses* **25**, 1-119 (2010).

[KLMS09] W. König, H. Lacoin, P. Mörters and N. Sidorova, A two cities theorem for the parabolic Anderson model *A*nn. Probab. **37:1**, 347–392 (2009).

[KS11] W. König and S. Schmidt, The parabolic Anderson model with acceleration and deceleration, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[LM11] H. Lacoin and P. Mörters, A scaling limit theorem for the parabolic Anderson model with exponential potential, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[M94] S. Molchanov, Lectures on random media. In: D. Bakry, R.D. Gill, and S. Molchanov, *Lectures on Probability Theory,* Ecole d'Eté de Probabilités de Saint-Flour XXII-1992, LNM 1581, pp. 242–411. Berlin, Springer (1994).

[MR94] S. Molchanov and A. Ruzmaikin, Lyapunov exponents and distributions of magnetic fields in dynamo models. In *The Dynkin Festschrift: Markov Processes and their Applications.* (Ed. Mark Freidlin) pp. 287–306, Birkhäuser (1994).

[M11] P. Mörters, The parabolic Anderson model with heavy-tailed potential, To appear in *Surveys in Stochastic Processes*, Proceedings of the 33rd SPA Conference in Berlin, 2009. Edited by J. Blath, P. Imkeller, and S. Rœlly. EMS Series of Congress Reports. (2011).

[MOS11] P. Mörters, M. Ortgiese and N. Sidorova, Ageing in the parabolic Anderson model, Supplement: Animation of limiting process, *Ann. Inst. Henri Poincaré, Probab. Stat.*, to appear (2011).

[P99] T. Povel, Confinement of Brownian motion among Poissonian obstacles in R^d, $d \geq 3$, *Probab. Theory Related Fields* **114:2**, 177–205 (1999).

[S11] B. Schmidt, On a semilinear variational problem, *ESAIM Control Optim. Calc. Var.* **17**, 86 - 101 (2011).

[S10] S. Schmidt, Das parabolische Anderson-Modell mit Be- und Entschleunigung (German), PhD thesis, University of Leipzig (2010). .

[SW11] A. Schnitzler and T. Wolff, Precise asymptotics for the parabolic Anderson model with a moving catalyst or trap, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[S91] A.-S. Sznitman, On the confinement property of two-dimensional Brownian motion among Poissonian obstacles, *Comm. Pure Appl. Math.* **44**, 1137–1170 (1991).

[S98] A.-S. Sznitman, *Brownian motion, Obstacles and Random Media.* Springer Berlin (1998).

[CGM11] F. Castell, O. Gün, and G. Maillard, Parabolic Anderson Model with a finite number of moving catalysts, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[DGRS11] A. Drewitz, J. Gärtner, A. Ram´irez, and R. Sun, Survival probability of a random walk among a Poisson system of moving traps, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[GH06] J. Gärtner, M. Heydenreich, Annealed asymptotics for the parabolic Anderson model with a moving catalyst, *Stoch. Process. Appl.* **116:11**, 1511–1529 (2006).

[GdH06] J. Gärtner, F. den Hollander, Intermittency in a catalytic random medium, *Ann. Probab.* **34:6**, 2219–2287 (2006).

[GdHM07] J. Gärtner, F. den Hollander, and G. Maillard, Intermittency on catalysts: symmetric exclusion, *Elec. J. Prob.* **12**, 516–573 (2007).

[GdHM09a] J. Gärtner, F. den Hollander, and G. Maillard, Intermittency on catalysts: three-dimensional simple symmetric exclusion, *Elec. J. Prob.* **14**, 2091–2129 (2009).

[GdHM09b] J. Gärtner, F. den Hollander and G. Maillard, Intermittency on catalysts, in: J. Blath, P. Mörters and M. Scheutzow (eds.), *Trends in Stochastic Analysis*, London Mathematical Society Lecture Note Series 353, Cambridge University Press, Cambridge, pp. 235–248 (2009).

[GdHM10] J. Gärtner, F. den Hollander, and G. Maillard, Intermittency on catalysts: voter model, *Ann. Probab.* **38:5**, 2066–2102 (2010).

[GdHM11] J. Gärtner, F. den Hollander, and G. Maillard, Quenched Lyapunov exponent for the parabolic Anderson model in a dynamic random environment, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[KS03] H. Kesten, V. Sidoravicius, Branching random walk with catalysts, *Elec. J. Prob.* **8**, 1–51 (2003).

[MMS11] G. Maillard, T. Mountford, and S. Schöpfer, Parabolic Anderson model with voter catalysts: dichotomy in the behavior of Lyapunov exponents, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).

[MZ11] S. Molchanov, H. Zhang, Parabolic Anderson model with the long range basic Hamiltonian and Weibull type random potential, To appear in *Probability in Complex Physical Systems*, Festschrift for Erwin Bolthausen and Jürgen Gärtner, Springer, Heidelberg (2011).