Lecture 5

Gaussian comparison inequalities

In our discussion of the DGFF and intermediate level sets thereof, we have so far managed to avoid technical facts on Gaussian processes. However, our later derivations will require these facts and so let us address these now. In this lecture we will focus on Gaussian comparison inequalities, starting with Kahane's inequality and its corollaries called the Slepian Lemma and Sudakov-Fernique inequality. We also show an application of Kahane's inequality to uniqueness of the Gaussian Multiplicative Chaos.

5.1. Kahane's inequality

In his development of the theory of Gaussian multiplicative chaos, Kahane made convenient use of inequalities that, generally, give comparison estimates of expectation of functions (usually convex in appropriate sense) of Gaussian random variables for two Gaussian vectors whose covariances can be compared pair by pair. One version of this inequality is as follows:

Proposition 5.1 [Kahane's Inequality] *Let X, Y be centered Gaussian vectors on $\mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n)$ a function whose second derivatives have subgaussian growth. Assume*

$$\forall i,j = 1,\dots,n: \quad \begin{cases} E(Y_iYj) > E(X_iX_j) & \Rightarrow \quad \dfrac{\partial f}{\partial x_i x_j}(x) \geq 0, \quad x \in \mathbb{R}^n \\ \\ E(Y_iYj) < E(X_iX_j) & \Rightarrow \quad \dfrac{\partial f}{\partial x_i x_j}(x) \leq 0, \quad x \in \mathbb{R}^n \end{cases} \tag{5.1}$$

Then

$$Ef(Y) \geq Ef(X). \tag{5.2}$$

A (multivariate) Gaussian vector X is said to be centered if it has vanishing expectation. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ has subgaussian growth if for each $\epsilon > 0$ there is $C(\epsilon)$ such that $|f(x)| \leq C(\epsilon)\mathrm{e}^{-\epsilon|x|^2}$ holds for all $x \in \mathbb{R}^n$. Note also for pairs i, j such that $E(Y_iY_j) = E(X_iX_j)$ the sign of $\frac{\partial f}{\partial x_i x_j}$ is not constrained above. For the proof we will need the following standard fact:

Lemma 5.2 [Gaussian integration by parts] *Let X be a centered Gaussian vector on $\mathbb{R}^n$ and $f \in C^1(\mathbb{R}^n)$ with ∇f having subgaussian growth. Then for any $i = 1, \dots, n$*

$$E\big(X_i f(X)\big) = \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) E\Big(\frac{\partial f}{\partial x_j}(X)\Big). \tag{5.3}$$

Exercise 5.3 *Prove the above lemma. Hint: The proof is elementary for X one-dimensional. The n-dimensional case can be reduced to this by writing $X_j = (C_{ij}/C_{ii})X_i + Z_j$ where $C_{jk} = \mathrm{Cov}(X_j X_k)$ and where $Z = (Z_1, \dots, Z_n)$ is independent of X_i on RHS.*

Gaussian integration by parts is the basis of:

Exercise 5.4 [Wick pairing formula] *Let $(X_1, \dots, X_{2n})$ be centered multivariate Gaussian (with some variables possibly repeating). Show that*

$$E(X_1 \dots X_{2n}) = \sum_{\pi:\ \text{pairing}} \prod_{i=1}^{n} \mathrm{Cov}\big(X_{\pi_1(i)} X_{\pi_2(i)}\big) \tag{5.4}$$

where π being a pairing means that π is a partition of the form

$$\{1, \dots, 2n\} = \bigcup_{i=1}^{n} \{\pi_1(i), \pi_2(i)\} \tag{5.5}$$

labeled so that $\pi_1(i) < \pi_2(i)$ for each $i = 1, \dots, n$ and $\pi_1(1) < \pi_1(2) \dots < \pi_1(n)$. (Note that $\pi_1(1)$ is always 1.)

The pairing formula plays an important role in computations involving Gaussian fields; in fact, it is the basis of perturbation calculations of functionals of Gaussian processes and organization of terms into Feynman diagrams.

Proof of Proposition 5.1. Suppose that X and Y are realized on the same probability space such that $X \perp\!\!\!\perp Y$. Define

$$Z_t := \sqrt{1-t^2}\, X + tY, \qquad t \in [0,1]. \tag{5.6}$$

Then $Z_0 = X$ and $Z_1 = Y$ and so

$$Ef(Y) - Ef(X) = \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} Ef(Z_t) \tag{5.7}$$

Using elementary calculus along with the above lemma,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ef(Z_t) &= \sum_{i=1}^{n} E\bigg(\Big(\frac{-t}{\sqrt{1-t^2}} X_i + Y_i\Big) \frac{\partial f}{\partial x_i}(Z_t)\bigg) \\ &= t \sum_{i,j=1}^{n} E\bigg(\big(E(Y_i Y_j) - E(X_i X_j)\big) \frac{\partial^2 f}{\partial x_i \partial x_j}(Z_t)\bigg) \end{aligned} \tag{5.8}$$

Based on our assumption, the expression in the expectation is non-negative for every realization of Z_t. Using this in (5.7) yields the claim. □

5.2. Kahane's theory of Gaussian Multiplicative Chaos

We will find this inequality useful later but Kahane's specific interest in Gaussian Multiplicative Chaos actually required a version that is not directly obtained from the above. Let us recall the setting more closely.

Let $D \subset \mathbb{R}^d$ be a bounded open set and let ν be a finite Borel measure on D. Assume that $C\colon D \times D \to \mathbb{R} \cup \{\infty\}$ is a symmetric, positive definite kernel in $L^2(\nu)$; i.e.,

$$\int_{D\times D} C(x,y) f(y) f(x) \nu(\mathrm{d}x) \geq 0 \tag{5.9}$$

holds for every bounded measurable $f\colon D \to \mathbb{R}$. If C is finite everywhere, then one can define a Gaussian process $\varphi = \mathcal{N}(0, C)$. Our interest is however in the situation when C is allowed to diverge on the diagonal $\{(x,x)\colon x \in D\} \subset D \times D$ which means that the Gaussian process exists only in a generalized sense — e.g., as a random distribution on a suitable space of test functions.

We will not try to specify the conditions on C that would make this fully meaningful; instead, we will just assume that C can be written as

$$C(x,y) = \sum_{k=1}^{\infty} C_k(x,y), \qquad x, y \in D, \tag{5.10}$$

where C_k is a finite covariance kernel for each k and the sum converges pointwise everywhere (including, possibly, to infinity when $x = y$). We then consider Gaussian processes

$$\varphi_k = \mathcal{N}(0, C_k) \quad \text{with} \quad \{\varphi_k\colon k \geq 1\} \text{ independent}. \tag{5.11}$$

Letting

$$\Phi_n(x) := \sum_{k=1}^{n} \varphi_k(x) \tag{5.12}$$

we define

$$\mu_n(\mathrm{d}x) := \mathrm{e}^{\Phi_n(x) - \frac{1}{2}\mathrm{Var}[\Phi_n(x)]} \nu(\mathrm{d}x). \tag{5.13}$$

Lemma 2.16 (or rather its proof) gives the existence of a random Borel measure μ_∞ such that for each $A \subset D$ Borel,

$$\mu_n(A) \underset{n\to\infty}{\longrightarrow} \mu_\infty(A) \quad \text{a.s.} \tag{5.14}$$

As the covariances $\mathrm{Cov}(\Phi_n(x), \Phi_n(y))$ converge to $C(x,y)$, we take μ_∞ as our interpretation of the measure

$$\text{“}\ \mathrm{e}^{\Phi_\infty(x) - \frac{1}{2}\mathrm{Var}[\Phi_\infty(x)]} \nu(\mathrm{d}x)\ \text{”} \tag{5.15}$$

for Φ_∞ being the centered generalized Gaussian field with covariance C. A key problem that Kahane faced was the issue of dependence of the limit measure on the above construction, and the uniqueness of the law of μ_∞ in general. This is, at least partially, resolved in:

Theorem 5.5 [Kahane's Uniqueness Theorem] *For $D \subset \mathbb{R}^d$ bounded and open, suppose there are covariance kernels $C_k, \widetilde{C}_k \colon D \times D \to \mathbb{R}$ such that*

(1) both C_k and $\widetilde{C}_k$ is continuous and non-negative everywhere on $D \times D$,

(2) for each $x, y \in D$,

$$\sum_{k=1}^{\infty} C_k(x,y) = \sum_{k=1}^{\infty} \widetilde{C}_k(x,y) \tag{5.16}$$

with, possibly, both these sums simultaneously infinite, and

(3) $\varphi_k = \mathcal{N}(0, C_k)$ and $\widetilde{\varphi}_k = \mathcal{N}(0, \widetilde{C}_k)$ have continuous paths a.s. for each $k \geq 1$.

Define, via (5.12–5.14), measures μ_∞ and $\widetilde{\mu}_\infty$ associated with these fields. Then

$$\mu_\infty(\mathrm{d}x) \overset{\text{law}}{=} \widetilde{\mu}_\infty(\mathrm{d}x). \tag{5.17}$$

In order to prove this we will need the following version of Proposition 5.1:

Proposition 5.6 [Kahane's convexity inequality] *Let $D \subset \mathbb{R}^n$ be bounded, open and let ν be a finite Borel measure on D. Let $C, \widetilde{C} \colon D \times D \to \mathbb{R}$ be covariances such that $\varphi = \mathcal{N}(0, C)$ and $\widetilde{\varphi} = \mathcal{N}(0, \widetilde{C})$ have continuous paths a.s. If*

$$\widetilde{C}(x,y) \geq C(x,y), \quad x, y \in D, \tag{5.18}$$

then for each convex $F \colon [0, \infty) \to \mathbb{R}$ with at most polynomial growth at infinity,

$$E\,F\Big(\int_D \mathrm{e}^{\widetilde{\varphi}(x) - \frac{1}{2}\mathrm{Var}[\widetilde{\varphi}(x)]} \nu(\mathrm{d}x)\Big) \geq E\,F\Big(\int_D \mathrm{e}^{\varphi(x) - \frac{1}{2}\mathrm{Var}[\varphi(x)]} \nu(\mathrm{d}x)\Big) \tag{5.19}$$

Proof. By approximation we may assume that $F \in C^2(\mathbb{R})$ (still convex). By the assumption of the continuity of the fields, it suffices to prove this for ν being a weighted sum of a finite number of Dirac masses, $\nu = \sum_{i=1}^n p_i \delta_{x_i}$ where $p_i > 0$. (The general case then follows by the weak limit of such measures to ν.)

Assume that the fields φ and $\widetilde{\varphi}$ are realized on the same probability space so that $\varphi \perp\!\!\!\perp \widetilde{\varphi}$. As before, set

$$\varphi_t(x) := \sqrt{1 - t^2}\,\varphi(x) + t\widetilde{\varphi}(x). \tag{5.20}$$

Since $\varphi_0(x) = \varphi(x)$ and $\varphi_1(x) = \widetilde{\varphi}(x)$, it suffices to show

$$\frac{\mathrm{d}}{\mathrm{d}t} E\,F\Big(\sum_{i=1}^n p_i \mathrm{e}^{\varphi_t(x_i) - \frac{1}{2}\mathrm{Var}[\varphi_t(x_i)]}\Big) \geq 0\,. \tag{5.21}$$

For this we abbreviate $W_t(x) := \mathrm{e}^{\varphi_t(x) - \frac{1}{2}\mathrm{Var}[\varphi_t(x)]}$ and use elementary calculus to get

$$\begin{aligned} E\,F\Big(\sum_{i=1}^n p_i W(x_i)\Big) = \sum_{i=1}^n p_i E\bigg(\Big[-\frac{t}{\sqrt{1-t^2}}\varphi(x_i) + \widetilde{\varphi}(x_i) \\ + t\mathrm{Var}\big(\varphi(x_i)\big) - t\mathrm{Var}\big(\widetilde{\varphi}(x_i)\big)\Big] W(x_i) F'(\cdots)\bigg) \end{aligned} \tag{5.22}$$

Next we integrate by parts the terms involving $\varphi(x_i)$, which results in the $\varphi(x_j)$-derivative of $W(x_i)$ or $F(\cdots)$. A similar process is applied to the term $\widetilde{\varphi}(x_i)$. A key point is that the contribution stemming from differentiation of $W(x_i)$ exactly cancels that coming from the variances. Hence we get

$$\begin{aligned} E\,F\Big(\sum_{i=1}^n p_i W(x_i)\Big) \\ = \sum_{i,j=1}^n p_i p_j \big[\widetilde{C}(x_i,x_j) - C(x_i,x_j)\big]\, E\Big(W(x_i)W(x_j)F''(\cdots)\Big). \end{aligned} \tag{5.23}$$

As $F'' \geq 0$, $W(x) \geq 0$ and $p_i, p_j \geq 0$, the assumption in (5.18) indeed implies (5.21). The claim follows by integration with respect to t. □

This permits us to give:

Proof of Theorem 5.5. It suffices to show that

$$\int f(x)\mu_\infty(\mathrm{d}x) \overset{\text{law}}{=} \int f(x)\widetilde{\mu}_\infty(\mathrm{d}x) \tag{5.24}$$

for any continuous $f\colon D \to [0,\infty)$ with compact $A := \operatorname{supp}(f)$. Let $\{C_k\colon k \geq 1\}$ and $\{\widetilde{C}_k\colon k \geq 1\}$ be the covariances in the statement. We claim that, for each $\epsilon > 0$ and each $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that

$$\sum_{k=1}^m C(x,y) < \epsilon + \sum_{k=1}^n \widetilde{C}(x,y), \qquad x,y \in A. \tag{5.25}$$

Indeed, letting F_m be the set of pairs $(x,y) \in A \times A$ where this inequality fails, the continuity of the covariances implies that F_m is closed (and thus compact) and decreasing with m. Moreover, (5.16) implies $\bigcap_{m\geq 1} F_m = \emptyset$. By Heine-Borel, we must have $F_m = \emptyset$ for m large enough thus giving us (5.25).

Interpreting the ϵ term on the right-hand side of (5.25) as the variance of the random variable $Z_\epsilon = \mathcal{N}(0,\epsilon)$ that is independent of $\widetilde{\varphi}$, Proposition 5.6 with the choice $F(x) := \mathrm{e}^{-\lambda x}$ for some $\lambda \geq 0$ gives us

$$E\big(\mathrm{e}^{-\lambda\, \mathrm{e}^{Z_\epsilon - \epsilon/2} \int f \mathrm{d}\widetilde{\mu}_m}\big) \geq E\big(\mathrm{e}^{-\lambda \int f \mathrm{d}\mu_n}\big)\,. \tag{5.26}$$

Invoking the limit (5.14) and taking $\epsilon \downarrow 0$ afterwards yields

$$E\big(\mathrm{e}^{-\lambda \int f \mathrm{d}\widetilde{\mu}_\infty}\big) \geq E\big(\mathrm{e}^{-\lambda \int f \mathrm{d}\mu_\infty}\big)\,. \tag{5.27}$$

By symmetry of the argument, equality holds here and since this is true for every $\lambda \geq 0$, we get (5.24) as desired. □

We remark that (as noted above) uniqueness of the GMC measure is now proved in a completely general setting in the recent work by Shamov.

5.3. Comparisons for the maximum

Our next task is to use Kahane's inequality from Theorem 5.1 to provide comparisons between the maxima of two Gaussian vectors with point-wise ordered covariances. We begin with a corollary to Theorem 5.1:

Corollary 5.7 *Suppose that X and Y are centered Gaussians on $\mathbb{R}^n$ such that*

$$E(X_i^2) = E(Y_i^2), \qquad i = 1, \dots, n \tag{5.28}$$

and

$$E(X_i X_j) \le E(Y_i Y_j), \qquad i, j = 1, \dots, n. \tag{5.29}$$

Then for any $t_1, \dots, t_n \in \mathbb{R}$,

$$P\big(X_i \le t_i \colon i = 1, \dots, n\big) \le P\big(Y_i \le t_i \colon i = 1, \dots, n\big). \tag{5.30}$$

Proof. Consider any collection $g_1, \dots, g_n \colon \mathbb{R} \to \mathbb{R}$ of non-negative bounded functions that are smooth and non-increasing. Define

$$f(x_1, \dots, x_n) := \prod_{i=1}^n g_i(x_i). \tag{5.31}$$

Then $\frac{\partial^2 f}{\partial x_i \partial x_j} \ge 0$ for each $i \ne j$. Hence, by Theorem 5.1, conditions (5.28–5.29) imply $Ef(Y) \ge Ef(X)$. From here the claim follows by letting g_i decrease to $1_{(-\infty, t_i]}$. □

From here we now immediately get:

Corollary 5.8 [Slepian's lemma] *Suppose X and Y are centered Gaussians on $\mathbb{R}^n$ with*

$$E(X_i^2) = E(Y_i^2), \qquad i = 1, \dots, n \tag{5.32}$$

and

$$E\big((X_i - X_j)^2\big) \le E\big((Y_i - Y_j)^2\big), \qquad i, j = 1, \dots, n. \tag{5.33}$$

Then for each $t \in \mathbb{R}$,

$$P\Big(\max_{i=1,\dots,n} X_i > t\Big) \le P\Big(\max_{i=1,\dots,n} Y_i > t\Big). \tag{5.34}$$

Proof. Set $t_1 = \dots = t_n := t$ in the previous corollary. □

Slepian's lemma has a nice verbal formulation using the following concept: Given a Gaussian process $\{X_t \colon t \in T\}$ on a set T, then

$$\rho_X(t, s) := \sqrt{E\big((X_t - X_s)^2\big)} \tag{5.35}$$

defines a pseudometric on T. Disregarding the prefix "pseudo", we will call this the *canonical*, or *intrinsic*, metric associated with Gaussian processes. Slepian's lemma can then be verbalized as follows: *For two process with the same variances, the one with larger intrinsic distances has stochastically larger maximum.*

Exercise 5.9 *Verify that ρ_X is indeed a pseudo-metric on T.*

The requirement of equal variances is actually often quite inconvenient as that requires adding suitable independent random variables to one, or both, Gaussians. It turns out, this inconvenience is removed when we restrict attention to the comparison of expected maxima:

Proposition 5.10 [Sudakov-Fernique inequality] *Suppose X and Y are centered Gaussians in $\mathbb{R}^n$ such that*

$$E\big((X_i - X_j)^2\big) \le E\big((Y_i - Y_j)^2\big), \qquad i,j = 1,\dots,n. \tag{5.36}$$

Then

$$E\big(\max_{i=1,\dots,n} X_i\big) \le E\big(\max_{i=1,\dots,n} Y_i\big) \tag{5.37}$$

Before we start the proof, notice a simple consequence of this inequality:

Exercise 5.11 *Show that, for any centered Gaussians $X_1,\dots,X_n$,*

$$E\big(\max_{i=1,\dots,n} X_i\big) \ge 0. \tag{5.38}$$

Prove that equality occurs if and only if $X_i = X_1$ for all $i = 1,\dots,n$ a.s.

Proof of Proposition 5.10. Consider the function

$$f_\beta(x_1,\dots,x_n) := \frac{1}{\beta}\log\Big(\sum_{i=1}^n \mathrm{e}^{\beta x_i}\Big). \tag{5.39}$$

For readers familiar with statistical mechanics, this quantity is a free energy of sorts. Hölder's inequality implies that $x \mapsto f_\beta(x)$ is convex (and smooth). In addition, we also get

$$\lim_{\beta\to\infty} f_\beta(x) = \max_{i=1,\dots,n} x_i. \tag{5.40}$$

Using dominated convergence, it therefore suffices to show that

$$Ef_\beta(X) \le Ef_\beta(Y), \qquad \beta \in (0,\infty). \tag{5.41}$$

The proof of this inequality will be based on a re-run of the proof of Kahane's inequality. Assuming again $X \perp\!\!\!\perp Y$ and letting $Z_t := \sqrt{1-t^2}\,X + tY$, differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}Ef_\beta(Z_t) = t\sum_{i,j=1}^n E\Big(\big[E(Y_iY_j) - E(X_iX_j)\big]\frac{\partial^2 f_\beta}{\partial x_i \partial x_j}(Z_t)\Big). \tag{5.42}$$

Now

$$\frac{\partial f_\beta}{\partial x_i} = \frac{\mathrm{e}^{\beta x_i}}{\sum_{j=1}^n \mathrm{e}^{\beta x_j}} =: p_i(x) \tag{5.43}$$

where we note that $p_i \ge 0$ with $\sum_{i=1}^n p_i(x) = 1$. For the second derivatives we get

$$\frac{\partial^2 f_\beta}{\partial x_i \partial x_j} = \beta\big[p_i(x)\delta_{ij} - p_i(x)p_j(x)\big]\,. \tag{5.44}$$

Plugging this on the right of (5.42) (and omitting the arguments of the second derivative as well as the p_i's) we then observe

$$\begin{aligned}
\sum_{i,j=1}^{n} \big[E(Y_iY_j) - E(X_iX_j)\big] \frac{\partial^2 f_\beta}{\partial x_i \partial x_j}& \\
&= \beta \sum_{i,j=1}^{n} \big[E(Y_iY_j) - E(X_iX_j)\big] \big[p_i\delta_{ij} - p_ip_j\big] \\
&= \beta \sum_{i,j=1}^{n} \big[E(Y_i^2) + E(X_i^2)\big] p_ip_j + \beta \sum_{i,j=1}^{n} \big[E(Y_iY_j) - E(X_iX_j)\big] p_ip_j \\
&= \frac{1}{2}\beta \sum_{i,j=1}^{n} \Big[E\big((Y_i - Y_j)^2\big) - E\big((X_i - X_j)^2\big)\Big] p_ip_j,
\end{aligned} \tag{5.45}$$

where we used that $\{p_i \colon i = 1, \dots, n\}$ are probabilities in the second line and then symmetrized the first sum under the exchange of i for j to wrap the contributions into the form on the third line. Invoking (5.36), this is non-negative (pointwise) and so we get (5.41) by integration. The claim follows. $\square$

The Sudakov-Fernique inequality can be verbalized as follows: *For two Gaussian processes, the one with larger intrinsic distances has larger expected maximum.*

Here is another, quite simple, inequality of this kind:

Exercise 5.12 *Suppose that X, resp., Y are centered Gaussian vectors on $\mathbb{R}^n$ with covariances C, resp., $\widetilde{C}$. Show that if $\widetilde{C} - C$ is positive semi-definite, then* (5.37) *holds.*